Perspectives on Pedagogical Content Knowledge in the Senior Secondary Mathematics Classroom

by

Nicole Maher, B.Ed (Hons)

Thesis Submitted in Partial Fulfilment

of the Requirements for the Degree of

Doctor of Philosophy

University of Tasmania

March 2019

Declaration of Originality

This thesis contains no material which has been accepted for a degree or diploma by the University or any other institution, except by way of background information and duly acknowledged in the thesis, and to the best of my knowledge and belief it contains no material previously published or written by another person except where due acknowledgement is made in the text or the thesis, nor does the thesis contain any material that infringes copyright.

08/03/2019

List of Publications related to this thesis

Maher, N., Chick, H., & Muir, T. "I believe the most helpful thing was him skipping over the proof": Examining PCK in a senior secondary mathematics lesson. In B. White, M. Chinnappan, & S. Trenholm (Eds.). *Opening up mathematics education research: Proceedings of the 39th annual conference of the Mathematics Education Research Group of Australiasia* (pp. 421-428). Adelaide: MERGA.

Maher, N., Muir, T., & Chick, H. (2015). Examining PCK in a senior secondary mathematics lesson. In M. Marshman, V. Geiger, & A. Bennison (Eds.). *Mathematics education in the margins*: *Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* (pp. 389-396). Sunshine Coast: MERGA.

Maher, N., Muir, T., & Chick, H. (2015). Secondary mathematics students' perceptions of their teachers' pedagogical content knowledge for teaching aspects of probability. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (pp. 4-18). Hobart: PME.

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08/03/2019

Statement of Ethical Conduct

The research associated with this thesis abides by the international and Australian codes on human and animal experimentation, the guidelines by the Australian Government's Office of the Gene Technology Regulator, and the rulings of the Safety, Ethics and Institutional Biosafety Committees of the University.

08/03/2019

Abstract

Pedagogical Content Knowledge (PCK) is an essential and complex facet of mathematics teacher knowledge that impacts on teaching and learning. Drawing upon Shulman's (1986) early conceptualisation and subsequent research (e.g., Ball, Thames, & Phelps, 2008; Chick, Baker, Pham, & Cheng, 2006; Magnusson, Krajcik, & Borko, 1999; Rowland, Huckstep, & Thwaites, 2005), PCK may be defined as the intricate blend of subject matter knowledge and other aspects of teacher knowledge (e.g., knowledge of curriculum and assessment). This study focused on PCK within a senior secondary mathematics teaching and learning context, by examining the ways in which mathematics teachers convey advanced mathematics ideas to students. The research investigation was designed to explore PCK, as a complex social phenomenon, in a multi-dimensional way. A qualitative research approach in the form of a collective case study, was used to generate and analyse data showing evidence of PCK from multiple sources and perspectives. Three senior secondary mathematics teachers and their Year 11/12 Mathematics Methods students, from two different schools in Northern Tasmania, took part in the study. Data generation methods included observation and video recording of 18 lessons (six per class), post-lesson interviews with the three teachers, as well as focus-group interviews and short written reflections from participating students. These methods allowed the researcher to obtain evidence of the teachers' enacted PCK through: observation, supplementary insights from the teachers' own perspective, and the students' interpretation of the PCK demonstrated by their teachers. Data were analysed for evidence of elements of PCK including those defined in the literature relating to mathematics teacher knowledge (e.g., Chick et al., 2006; Rowland et al., 2005). The results were

qualitatively described, depicting the different elements of PCK in action in the classroom and offering insight into this knowledge from the perspectives of the teachers and their students.

Findings suggest that multiple and interconnected aspects of PCK were enacted by the teachers in ways that focused on the teaching and learning of mathematics procedures with an emphasis on solving standard text-book exercises. Elements of PCK including *knowledge of examples*, *teacher demonstration*, *knowledge of student errors*, and *anticipation of complexity* were particularly evident in the data. The teachers' justification for their own instructional choices and actions, evident in their post-lesson interview responses, enhanced the depth and quality of evidence of PCK. Their comments reflected a perception of the constraints of the context of the Mathematics Methods syllabus, particularly in relation to the high stakes external examination. The teachers made pragmatic decisions about what to teach and how to teach it, particularly when unexpected situations arose where teachers had to call upon their own mathematical content knowledge in-the-moment. In general, the teachers avoided addressing the deeper conceptual underpinnings of mathematical ideas in favour of solving standard text-book questions. The students noticed and appreciated aspects of their teacher's PCK, particularly those relating to explicating the steps involved in completing these questions.

This study contributes to the research into mathematics teachers' knowledge by exploring some of the complexities and tensions of PCK within the context of the senior secondary mathematics classroom, particularly in relation to the ways in which perceived contextual constraints impact upon this knowledge and its growth.

Dedication

This thesis is dedicated to my parents Sue and Eric, for their immeasurable support throughout my candidature and always.

Acknowledgments

I wish to sincerely thank my supervisors Associate Professor Tracey Muir and Associate Professor Helen Chick. Tracey, thank you for your continuous support, guidance, patience and expertise throughout the research process and in the shaping of the chapters of the thesis. Your mentorship has played a significant role in all aspects of my candidature. I would like to thank Helen for her insightful counsel and for deepening my thinking about analysing the complexities of teaching and learning senior secondary mathematics, within the context of this study.

I express my heartfelt gratitude to my mother Sue for walking alongside me throughout this PhD journey, from the initial stages of listening to my ideas and reading passages (usually under the filename 'the thing') to the countless hours of proofreading chapters. Thank you for guiding my writing in ways that have inspired me to strive for depth and clarity. I am deeply grateful for my father Eric whose continued support, both tangible and intangible, has allowed me to complete this thesis.

I wish to acknowledge Linda Bathish for her expertise, initiative, and tireless work on the formatting of all aspects of this thesis. Linda thank you for your care, attention to detail, and efficiency. It was such a pleasure to work with you.

It is important that I acknowledge the participants in this study. I am thankful for the trust they placed in me, and for their generosity of time, knowledge, and insight, all of which are at the heart of this thesis.

Finally, I thank my husband Robbie and my son Ryan for their patience and encouragement throughout my PhD, and for believing, even in the tougher times, that it would be worth it.

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Chapter 1

Introduction

1.1 Context of the Study

The nature of senior secondary mathematics education in Australia is a challenging space, particularly in relation to high-stakes external examinations and requirements associated with the Australian Tertiary Admissions Ranking (ATAR) score. Tensions relating to students' perception of the limited incentive to study higher level mathematic have impacted upon the enrolment numbers in these courses (e.g., Hine, 2018). For example, research suggests that some students choose to enrol in less rigorous mathematics courses that satisfy university entrance requirements and are perceived to increase the opportunity to achieve their ATAR goals (Forgasz, 2006; Hine, 2018). This situation is of concern because high-level mathematics is considered an essential component of advancement in research and innovation in a broad range of sectors (e.g., Ainley, Kos, & Nicholas, 2008; Hine, 2018).

Given the on-going concern about the level of student participation and achievement in rigorous pre-tertiary mathematics courses (e.g., Hine, 2018; Noyles & Sealey, 2012), teaching and learning at the senior secondary level is an area ripe for further research. In addition, research into the nature of the senior secondary mathematics classrooms supports and contributes to the current national focus on professional standards (Australian Institute for Teaching and School Leadership) by examining teacher knowledge.

The present study focuses on the enactment of teacher knowledge in three senior secondary classrooms over a period of six lessons per class. Data generated from classroom observations and interviews with the teachers and students, enabled the fine-grained analysis and discussion of the ways in which mathematics-related teacher knowledge plays out at the senior secondary level. Further detail on the study design is discussed in Section 1.4.

1.2 Pedagogical Content Knowledge

Effective teachers need knowledge of students' thinking, knowledge of mathematical content, and knowledge of how to represent the content so that it makes sense to others (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Krauss Baumert, & Blum, 2008). In other words, the knowledge needed to teach mathematics is multidimensional (Carpenter, Fennema, & Franke, 1996; Petrou & Goulding, 2011). Substantial progress has been made towards mapping out and identifying the constituent parts of mathematics teacher knowledge, often referred to as categories of knowledge (e.g., Ball et al., 2008; Chick, Baker, Pham, & Cheng, 2006; Rowland, Huckstep, & Thwaites, 2005). One such category, pedagogical content knowledge (PCK), is at the core of this investigation.

Pedagogical content knowledge (PCK) is concerned with the way subject matter is transformed from the knowledge held by the teacher into the content of instruction (Shulman, 1986). In one of his widely cited seminal papers, Shulman (1986) described PCK as an intricate blend of content and pedagogy that encompasses all that is needed to teach a subject or topic in a way that makes it comprehensible to others.

In the decades since Shulman's original conceptualisation of PCK, the notion of what constitutes all that is needed to make content accessible to the learner has broadened in both depth and scope (e.g., Chick & Beswick, 2017; Hashweh, 2005; Magnusson, Krajcik, & Borko, 1999). In a recent reflection on the genesis of PCK, Shulman (2015) acknowledged how far the concept of PCK had developed from its infancy to "a citizen of many countries" linked to "the normative needs" of the environment within which it is enacted (p. 20). This reflection points to the influence of the broader social and cultural context on the enactment of PCK.

It is also important to highlight that PCK is not a fixed body of knowledge held by the teacher (e.g., Fennema & Franke, 1992; Mason & Davis, 2013) but can develop in the moment-of-teaching through classroom interactions. This dynamic view of PCK also acknowledges the inherent interplay between categories of teacher knowledge (Hashweh, 2005). The process of examining the concept of PCK has led to a broadening of its description to encompass interactions between knowledge categories and the broader teaching and learning context.

1.3 Overview of Research into PCK

It is generally accepted in the field of mathematics education research that PCK impacts upon teaching and learning (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Hill et al., 2008). Research in relation to PCK has focused mainly on pre-service and practicing teachers in the context of teaching primary mathematics (e.g., Baker & Chick, 2006; McDuffie, 2004; Park & Oliver, 2008; Rowland et al., 2005). Comparatively few studies have focused on secondary mathematics education, as emphasised by Matthews (2013) in her review of the influence of PCK across grade bands. Even fewer studies (e.g., Potari, Zachariades, Christou, Kyriazis, & Pitta-Pantazi, 2007) have addressed the nature of teacher knowledge at the senior secondary level, which involves content including functions and their inverses, calculus, and probability distributions.

Recent studies examining PCK at the secondary level include the development of *The Knowledge of Algebra for Teaching framework* (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012) and the *Cognitive Activation in the Classroom* (COACTIV) project, an investigation into the professional competence of secondary mathematics teachers' PCK (Baumert et al., 2010; Krauss et al., 2008). Other studies, in both the primary and secondary contexts, have concentrated on specific aspects of teachers' practice which inform us about their PCK including the choice and use of examples (e.g., Chick, 2009; Huang, 2017; Zodik & Zaslavsky, 2008).

Few studies, however, have used teacher knowledge frameworks to explore the enactment of PCK at the secondary and senior secondary levels in the classroom (e.g., McCrory et al., 2012). Furthermore, research into students' perceptions of the PCK they consider to be helpful in assisting them with their learning of abstract mathematics, has been largely unexplored. The Learner Perspective Study (e.g., Anthony, Kaur, Ohtani, & Clarke, 2013), however, suggested that the student voice is central to any exploration of classroom practice. For example, Huang and Barlow (2013) explored the extent to which the students' and the teacher's perceptions of important lesson events correspond with each other. The authors found that the students in their study particularly noticed those lesson events that were intentionally designed by their teacher to help them to overcome difficulties or to highlight key aspects of the content.

1.4 Research Questions and Study Design

This study aims to investigate the nature of PCK for teaching senior mathematics and to examine this knowledge from the perspectives of the researcher, the teachers, and the students. To that end, the research questions are:

1. What aspects of mathematical pedagogical content knowledge are evident in the interactions between teachers and their students during the teaching and learning of senior secondary mathematics content?
2. What aspects of mathematical pedagogical content knowledge do teachers discuss and attribute their instructional decisions to when analysing their interactions with students during the teaching and learning of senior secondary mathematics content?
3. What aspects of mathematical pedagogical content knowledge are identified by students as having an impact on their learning of senior secondary mathematics content?

To answer these questions, qualitative research approaches were used to generate and analyse evidence of PCK in the senior secondary mathematics classroom. Given the complexity of classroom interactions, data were generated from multiple perspectives. Lesson observation and video recordings of the lessons provided rich and reviewable data ripe for detailed analysis (Mousley, 1998) from the researcher's perspective. In addition, post-lesson interviews with teachers, and focus-group and written responses from students, provided data for further analysis of aspects of PCK from their own points of view.

Data from the multiple sources and perspectives were analysed inductively and deductively using existing teacher knowledge frameworks, namely the Knowledge Quartet (KQ) (e.g., Rowland, et al., 2005) and the Chick et al. PCK framework (e.g., Chick et al, 2006). The results were qualitatively described, in the form of scenarios depicting evidence of PCK in action in the classroom and offering insight into this knowledge from the viewpoints of the teachers and their students.

1.5 The Researcher's Motivation for the Study

The initial motivation for this study was the researcher's own interest in the ways in which teachers of senior secondary mathematics make advanced mathematics content accessible for their students. As an experienced teacher of secondary mathematics (including some senior secondary mathematics courses), the researcher appreciated the complexity of making abstract mathematical ideas comprehensible to students. She became inspired, during a university secondment, by the idea of PCK as a powerful category of teacher knowledge, particularly within the context of her own professional growth.

1.6 The Australian Senior Secondary Curriculum: Mathematics

The expectations of Year 11 and 12 mathematics courses in Australia are defined by national standards developed by ACARA and directly linked to the *Australian Senior Secondary Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.). These standards are intended to be used by teachers to define expectations of course requirements including the course content and standards of assessment. The *Senior Secondary Australian*

Curriculum: Mathematics consists of four subjects (Essential Mathematics, General Mathematics, Mathematical Methods, and Specialist Mathematics). Each subject, organised into four units, focuses on a specific pathway designed to meet the learning needs of particular groups of students (ACARA, n.d.). Mathematical Methods is the subject upon which the course featured in this study (Mathematics Methods) is derived and was designed to "broaden students' mathematical experience and provide different scenarios for incorporating mathematical arguments and problem solving" (ACARA, n.d.). Mathematical Methods focuses on the topics of calculus and statistical analysis. The topic of calculus includes the use of functions, their derivatives and integrals, as well as modelling processes. The statistics component focuses on the phenomena of uncertainty and variation, including an introduction to statistical inference.

The course featured in this study, Mathematics Methods MTM315 (now revised to MTM415), was one of the two most demanding Year 11/12 pre-tertiary mathematics courses offered in Tasmanian schools and colleges. Mathematics Methods was designed for students intending to pursue tertiary pathways including the sciences, health sciences, engineering, and economics. While the course syllabus document acknowledged that the content is set out under topic headings (e.g., functions and graphs, circular functions, calculus, statistics and probability), a more integrated approach is recommended because much of the course content is inter-related (Tasmanian Qualifications Authority [TQA], 2014). In addition, the course syllabus document (TQA, 2014) suggested that ideas should be developed, where applicable, within the context of practical applications, with the aim of providing richer mathematical experiences as distinct from a collection of skills (TQA, 2014); "Students thereby have the opportunity to observe and make connections between related aspects of the course and the real world and to develop further some important abstract ideas" (TQA, 2014, p. 3).

The assessment of Mathematics Methods was criterion-based "a form of outcomes assessment that identifies the extent of student achievement at an appropriate end-point of study" (TQA, 2014, p. 8). Specific criteria were assessed using a combination of internal and external assessment approaches. The external component consisted of a high stakes examination supervised by the Tasmanian Qualifications Authority (now called Office of Tasmanian Assessment, Standards and Certification).The use of technology, including graphics calculators, CAS (computer algebra system) technologies, and computer software, were considered integral to the course, both for the development of concepts and as a tool for solving problems; in other words, both the functional and pedagogical use of CAS was recommended. The functional use of CAS refers to using the technology primarily to "produce answers" when performing routine procedures (e.g., Kendal & Stacey, 2001). The pedagogical use of CAS involves using the technology as a teaching tool to explore or develop a mathematical idea (2001).

The topics of focus in this study were calculus and probability. The calculus topic introduced students to applications of differential and integral calculus. Some of the key areas of focus included the development of skills (e.g., using rules for derivatives, integrals, and definite integrals), practical applications (e.g., optimisation problems), "limit theorems made plausible", and the informal treatment of the fundamental theorem of calculus (TQA, 2014, p. 6). The statistics component of Mathematics Methods involved constructing and interpreting discrete and continuous probability distributions. Typically, discrete random variables were introduced first, including the calculation and interpretation of the mean (expected value) and variance of discrete random variables, followed by the application of binomial and hypergeometric distributions to model discrete random processes.

1.7 Outline of the Thesis

This chapter introduced the context of the study including motivation from the researcher's perspective followed by a brief introduction to the literature relating to PCK. The significance of the study was addressed with an overview of the study design and, finally, a short description of senior secondary mathematics within the context of the Australian Curriculum.

Chapter 2 provides a review of the literature, including the multi-faceted nature of teacher knowledge, and the evolution of PCK as a complex and nuanced aspect of this knowledge. Key mathematics teacher knowledge frameworks and the ways in which they have been used to examine teaching are addressed. This is followed by a discussion of research in the context of the senior secondary mathematics classroom which highlights insights and opportunities for exploration in the present study.

Chapter 3 is the Methodology chapter, which addresses the design and implementation of the study, ethical considerations regarding the teacher and student participants, followed by a justification for and explanation of the data generation methods. Issues of the trustworthiness of this qualitative study are also discussed.

Chapter 4 presents the results of the study in the form of 14 scenarios each consisting of a teaching and learning event from a specific lesson, corresponding responses from the teacher and or students, and a commentary highlighting evidence of aspects of the teachers' PCK. Each scenario provides a detailed snapshot of evidence of senior secondary mathematics teachers' PCK and enables examination from multiple perspectives.

Chapter 5, the Discussion chapter, addresses the three research questions by drawing upon the results presented in the Scenarios described in Chapter 4. The findings from each of the three perspectives are then drawn together and key aspects of PCK evident in the interactions between the teachers and students in the senior secondary mathematics classroom are addressed.

Chapter 6 is the concluding chapter which provides an overview of the study's purpose and design, followed by a reflection on the research findings, their contribution to knowledge, and implications for current and future research.

Chapter 2

Literature Review

This chapter is structured around four broad areas, beginning with an introduction to the multi-faceted nature of teacher knowledge followed by a discussion of Shulman's seminal work on a professional knowledge base for teaching. A discussion of three mathematics teacher knowledge frameworks is then provided, including an examination of the ideas underpinning the components of these frameworks. The chapter concludes with a review of literature relating to the senior secondary mathematics context.

2.1 Overview of Teacher Knowledge

Teacher knowledge is a complex phenomenon comprising many different facets including knowledge of subject matter, knowledge of learners, and knowledge of teaching strategies. Of particular interest are the ways in which teachers draw upon and use this knowledge in their teaching. For example, teachers need knowledge of the content they are required to teach, yet the nature of this knowledge and how teachers engage with it in their practice, is not clearly understood (Hill et al., 2008; Petrou & Goulding, 2011). Furthermore, the correlation between content knowledge and teaching quality is strikingly weak, suggesting that providing teachers with more content knowledge in a given discipline does not necessarily translate into better teaching (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Shulman, 2015).

The elusive nature of the role of content knowledge in teaching underpins much of the research into teacher knowledge (e.g., Ball et al., 2008; Rowland et al, 2005; Shulman, 1986). In particular, Shulman's introduction of the idea of pedagogical content knowledge (PCK), in the 1980s, illuminated and paved the way for researchers to examine teacher knowledge in ways that seek to explain and deepen our understanding of the relationship between teachers' content and pedagogical knowledge. PCK is a particularly powerful construct for exploring teacher knowledge because it encompasses an intricate blend of both content knowledge and pedagogical knowledge – the kind of knowledge that is unique to teaching (Ball et al., 2008).

2.1.1 The Discipline Specific Nature of Teacher Knowledge

Over the past few decades there has been a profound shift in focus within the education research community from an emphasis on generic pedagogic issues such as general teaching strategies, classroom management, curriculum design, and assessment, towards a renewed concern for the subject matter itself and how it is taught (Marks, 1990; Shulman, 1986; Stodolsky & Grossman, 1995; Stylianides & Ball, 2008). This revived interest in subject matter evolved because an obvious separation between content and pedagogy had become apparent in teacher education research and policy making during the 1980s (Ball & Bass, 2000; Shulman, 1986, 1987). Education policies and teacher evaluation processes tended to treat teachers' subject knowledge and pedagogy as mutually exclusive domains (Shulman, 1986). For example, most indicators of teacher effectiveness involved generic, albeit important, classroom practices such as questioning techniques includingwait time and

Bloom's taxonomy (Bloom, 1956), rather than the adequacy and accuracy with which specific subject matter was taught (Shulman, 1986). Shulman referred to the notable absence in focus on subject matter knowledge as the "missing paradigm" and highlighted the necessity to examine the relationship between content knowledge and pedagogy by investigating how teachers draw on their expertise of subject matter in the process of teaching (1986, p. 7).

Shulman's work on the complex and multi-faceted nature of teacher knowledge is seminal in the field of educational research. Much of the literature on teacher knowledge, particularly from the late 1980s to the present, draws upon and/or builds on the work of Shulman and his colleagues (e.g., Ball et al., 2008; Chick et al., 2006; Hashweh, 2005; Hill et al., 2008; Krauss et al., 2008; Marks, 1990; Rowland et al., 2005).

2.2 Shulman's Knowledge Base for Teaching

The work of Shulman and his associates was influenced by reform proposals to professionalise teaching which were based on the premise that there exists a "codifiable" knowledge base for teaching (Shulman, 1987, p. 4). Shulman's approach to defining such a knowledge base focused on investigating evidence of the ways in which content knowledge and pedagogical strategies "interacted in the minds of teachers" (1987, p. 5). Of special interest was consideration of the differences in the way expert teachers delivered the same material that presented difficulties for novice teachers, in order to determine the kind of knowledge and skill required to teach demanding subject matter well (Shulman, 1987).

Shulman's initial formulation of seven categories of teacher knowledge (see Figure 2.1) emerged from the assumption that a rich and extensive knowledge base not only exists, but originates from plentiful and diverse sources, such as scholarship in content disciplines, the educational setting, education research, and the wisdom of practice itself (Shulman, 1987).

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organisation that appear to transcend subject matter.
- Knowledge of learners and their characteristics.
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures.
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.
- Content knowledge
- Curriculum knowledge, with particular grasp of materials and programs that serve as “tools of the trade” for teachers.
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding

Figure 2.1. Shulman’s categories of a knowledge base for teaching (Shulman, 1987, p. 8).

The first four categories listed in Figure 2.1 are generic aspects of teacher knowledge, while the final three categories relate specifically to teachers’ knowledge of content. Although Shulman acknowledged that codifying generic aspects of teacher knowledge plays a vital part in understanding the complexity of teaching, the final three categories in Figure 2.1 were of importance to the work of Shulman and his colleagues because they are specifically related to content. It is also important to emphasise that Shulman’s seven categories of teacher knowledge are not specific to one particular discipline (e.g., mathematics, science, history), but rather provide a broad framework upon which to consider a knowledge base for teaching in general. The following sections elaborate on each of the three content-related knowledge categories.

2.2.1 Content Knowledge

Shulman described content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (1986, p. 9). Drawing on Schwab's (1967) notion of substantive and syntactic knowledge structures, Shulman emphasised the importance of the way in which knowledge is held by teachers. That is, a teacher's knowledge of content should go beyond knowledge of the skills and concepts of a discipline, to include knowledge of the way those skills and concepts are organised and connected within the discipline (substantive knowledge) (Petrou & Goulding, 2011). In addition, knowledge of content includes an appreciation for the ways in which the knowledge is generated within a specific discipline and how it is deemed valid or otherwise (syntactic knowledge) (Petrou & Goulding, 2011).

2.2.2 Curriculum Knowledge

Curriculum knowledge was described by Shulman as knowledge of the full gamut of materials and resources, including text books, available for teaching a topic at a given year level as suggested in the following excerpt:

Represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials in relation to those programs, and the set of characteristics that serve as both the indications and the contraindications for the use of particular curriculum or program materials or particular circumstances. (Shulman, 1986, p. 10)

Shulman's description also suggests that curriculum knowledge encompasses a teacher's ability to critique the appropriateness of particular materials, and to know about, and draw upon, alternative resources for particular teaching and learning situations. Petrou and Goulding (2011) however, point out that Shulman's conceptualisation of curriculum knowledge implies a relatively free and flexible degree of choice of materials and approaches which may not be applicable in some contexts. As a case in point, Petrou and Goulding highlight the situation in the UK where contemporary education practice is constrained by "official guidance and assessment systems"; therefore, teachers may not choose to draw upon a full range of available resources – or even realise they are available – because they are limited by formal testing schedules (2011, p. 17).

2.2.3 Pedagogical Content Knowledge

The third and arguably the most influential content-related knowledge category is pedagogical content knowledge (PCK). Among the seven knowledge categories, Shulman highlighted the importance of PCK because it identifies and melds together the two classic bodies of knowledge for teaching: content and pedagogy. In fact, pedagogical content knowledge is the category most likely to distinguish "the understanding of the content specialist from that of the pedagogue" (Shulman, 1987, p. 4). Given the explanatory power of Shulman's original description of PCK, it is not surprising that excerpts from the description below have been often cited in mathematics education research literature (e.g., Hill et al., 2008; Marks, 1990; Meredith, 1995):

> Within the category of pedagogical content knowledge, I include, for the most regularly taught topics in one's subject area, the most useful

> forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations - in a word, the ways of representing and formulating the subject that make it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas others originate in the wisdom of practice.
>
> Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions, which they so often are, teachers need knowledge of strategies most likely to be fruitful in reorganising the understanding of learners, because those learners are unlikely to appear before them as blank slates (Shulman, 1986, p. 9).

The concept of PCK originated from a deepened awareness and appreciation for the discipline specific nature of teacher knowledge. In a recent reflective essay, Shulman (2015) acknowledged that the original conceptualisation of PCK was not intended to be fixed and final, but rather, provided a generative starting point for further development and refinement:

> Any idea must be understood as a contribution to the conversation of which it was part, not as a universal truth or generalization. PCK

> certainly has its weaknesses, and I trust that many of you are shoring up those deficiencies, elaborating and going beyond the initial formulation, as should be the fate of any reasonably good idea (p.11).

Shulman himself, identifies some limitations of the original conceptualisation of PCK (2015). For example, he highlights an absence of focus on affect and non-cognitive attributes, particularly given that teachers' actions are "connected to their own affective and motivation states, as well as their ability to influence the feelings, motives, persistence, and identity formation processes of their students" (2015, p.17). In retrospect, Shulman also acknowledges that PCK is inevitably situated within a broader teaching and learning context, and as such "the big idea within PCK was that all teaching must be mindfully situated in the disciplinary, cultural, personal, and social settings in which it occurs" (2015, p.17).

While Shulman's reflections on the limitations of his original formulation of PCK are powerful and interesting in their own right, they also give perspective and insight into the nature of some of the subsequent developments in relation to PCK. As noted by Graeber and Tirosh (2008), in their review of the ways in which other researchers have extended and elaborated on Shulman's conceptualisation of PCK, the overall trend has been to widen the definition of PCK. That is, PCK has come to encompass a broader range of knowledge types and attributes.

Marks (1990) offers a broad interpretation of the meaning of PCK, and how it is generated. According to Marks, there are three different ways in which PCK can be derived from other knowledge categories. Some aspects of PCK derive from subject matter and involve "an adaptation of subject knowledge for pedagogical purposes," such as the sequencing of examples for instruction (Marks, 1990, p. 7). Other aspects originate from a combination of content and pedagogical knowledge, such as knowledge of students' common errors or misconceptions (p. 11). In addition, Marks suggests there are aspects of PCK which derive from general pedagogical knowledge, including teachers' use of questioning strategies (p. 7). He unpacks the subtle nature of this derivation by explaining how generic pedagogical knowledge "engenders pedagogical content knowledge" through the process of "specification", the "appropriate instantiation of a broadly applicable idea in a particular context" (Marks, 1990, p. 8). Marks exemplifies this specification process by highlighting that pre-service teachers typically study pedagogical strategies such as questioning in generic terms, and then must apply these generic skills within the context of their particular subject area (1990, p. 8).

In addition, non-cognitive attributes, including beliefs, have been recognised by several scholars from a range of disciplines, as an integral part of PCK (e.g., Chick & Beswick, 2017; Magnusson et al, 1999), or as an influence on PCK (e.g., Gess-Newsome, 2015; Grossman, 1990; Hashweh, 2005). There is also increasing awareness of the influence of the broader classroom, school, and educational context as key determinants for teaching and learning (e.g., Grossman & Stodolsky, 1995; Shulman, 2015). Grossman describes knowledge of context as comprising the following:

> Knowledge of context includes: knowledge of the districts in which teachers work, including the opportunities, expectations, and constraints posed by the districts; knowledge of the school setting, including the school 'culture', departmental guidelines, and other contextual factors at the school level that affect instruction; and knowledge of specific

students and communities, and the students' backgrounds, families, particular strengths, weaknesses, and interests. (Grossman, 1990, p.9)

Of relevance to the present study is the idea that school subjects or specific courses serve as "context" for secondary teachers. Grossman and Stodolsky (1995) examined the notion of "content as context" in light of research into the contexts of secondary teaching (p. 5). Their findings highlighted that secondary teachers readily discussed their subjects as part of their everyday work with focus on the "constraints and possibilities they perceive as offered by specific school subjects" (p. 7). For example, secondary mathematics teachers tended to discuss the constraints of content and the demands of meeting the requirements of a sequential and well-established curriculum, within a limited time frame (p. 7). Several scholars, such as Grossman (1990), and those of whom have drawn upon her work (e.g., Hashweh, 2005; Magnusson et al., 1999) have acknowledged the influence of context on PCK.

2.2.4 Shulman's Cycle of Pedagogical Reasoning and Action

Another, possibly lesser known framework, developed by Shulman and his colleagues is the cycle of *Pedagogical Reasoning and Action.* This model is particularly worth highlighting because it describes teacher knowledge in action (including PCK) and underpins aspects of one of the teacher knowledge frameworks discussed in a later section. The cycle of pedagogical reasoning and action comprises six key aspects of teaching: *comprehension, transformation, instruction, evaluation, reflection*, and *new comprehension* (Shulman & Sykes, 1986; Wilson, Shulman, & Richert, 1987).

Pedagogical reasoning and action begin with *comprehension* as teachers need to understand the substantive and syntactic knowledge structure of the content they are required to teach (Wilson et al., 1987). Along with understanding the content itself, comprehension is also concerned with understanding the goals and purposes of the content in a broader context. The next stage in the cycle of pedagogical reasoning and action is *transformation*, during which teachers transform their own understanding of specific content knowledge in powerfully pedagogical ways for their students. Furthermore, according to Wilson and colleagues, "transformation of subject matter knowledge is at the heart of teaching in secondary schools" (1987, p. 117).

The *instruction* phase refers to teaching in action, the observable performance of the teacher such as: general management of the class, questioning, providing explanations, and eliciting discussion (e.g., Shulman, 1987; Wilson et al., 1987). *Evaluation* involves the ongoing informal processes of checking for understanding, as well as the more formal modes such as unit tests and examinations (Shulman, 1987). Teachers evaluate their own teaching through the process of *reflection* and ideally this leads to *new comprehension* (Shulman, 1987; Wilson et al., 1987).

2.2.5 Summary of the Section

This section examined the evolution of PCK, from the circumstances of its conception to issues relating to its complex nature. PCK is difficult to characterise definitively because it comprises an intricate blend of several knowledge types, and is context specific (e.g., Gess-Newsome, 2015; Marks, 1990). The sophisticated and multi-faceted nature of teacher knowledge underpins the value and purpose of frameworks which unpack teacher knowledge including PCK. It follows, then, that any theoretical model used to carefully examine a sophisticated and multifaceted practice such as teaching, should itself be complex and nuanced. The next section focuses on teacher knowledge frameworks which have specifically been used in mathematics education research.

2.3 Teacher Knowledge Frameworks in Mathematics Education

The previous section provided an overview and introduction to teacher knowledge of a generic nature. This section will explore a number of frameworks which have been specifically developed for defining and categorising mathematics teacher knowledge and examining the ways in which this knowledge is enacted in the classroom. Currently there is no widespread agreement on any one particular framework for adequately encapsulating and describing mathematics teacher knowledge (Petrou & Goulding, 2011). The following sections discuss three frameworks (Ball et al., 2008; Chick et al., 2006; Rowland & Turner, 2007) which have been used for analysing mathematics teacher knowledge and which were used as a basis for interpreting the work undertaken by teachers in this study.

2.3.1 Ball and Colleagues' Mathematical Knowledge for Teaching (MKT) Framework

The framework for Mathematics Knowledge for Teaching (MKT) developed by Ball and her colleagues (e.g., Ball et al., 2008; Hill et al., 2008), builds on Shulman's initial conceptualisation of subject matter knowledge (SMK) and PCK in important ways. MKT refers to the knowledge "needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399) and includes, but goes beyond, Shulman's notion of PCK (Herbst & Kosko, 2014). The rationale behind the development of the MKT framework was to generate empirical evidence for, and to further refine, Shulman's original theoretical distinction between SMK and PCK (e.g., Ball et al., 2008). Ball and her colleagues adopted a practice-based approach to identifying MKT particularly within the elementary (primary) and middle school contexts (Hill et al., 2008). The iconic domain map in Figure 2.2 depicts the way in which MKT refines Shulman's initial categories of SMK and PCK. The left side of the oval represents Shulman's original subject matter knowledge category and has been subdivided into *common content knowledge* (CCK), *specialised content knowledge* (SCK), and *knowledge to the mathematical horizon*. On the right side of the oval Shulman's concept of pedagogical content knowledge (PCK) is maintained but is further divided into *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). Shulman's original *curricular knowledge* category has been included as another subdomain of PCK and is referred to as *knowledge of content and curriculum* (KCC).

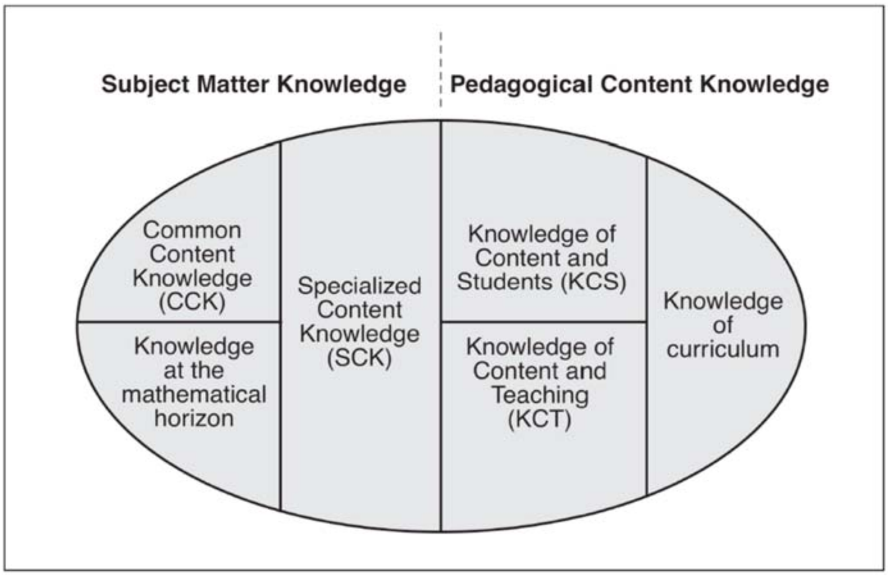


Figure 2.2. Domain map for Mathematics Knowledge for Teaching (Ball et al., 2008, p. 403)

Descriptions of each of the subdomains in Ball and her colleagues' domain map for Mathematical Knowledge for Teaching are given in Table 2.1.

Table 2.1

Definitions of each of the subdomains in the domain map for Mathematical Knowledge for Teaching (Ball et al., 2008)

Domain	Subdomain	Description of subdomain
	Common content knowledge (CCK)	CCK is the mathematical knowledge used in a variety of settings (i.e., not unique to teaching). CCK involves correctly solving mathematics problems.
Subject Matter Knowledge (SMK)	Specialised content knowledge (SCK)	SCK is mathematical knowledge unique to teaching. SCK deals with mathematics content in its decompressed (or unpacked) form in order to make the content accessible to students.
	Horizon knowledge	Horizon knowledge is an awareness of how mathematics topics span over the curriculum. This category has been provisionally included as another subdomain of SMK.
	Knowledge of content and students (KCS)	KCS is knowledge that combines knowing about mathematics and knowing about students (e.g., knowledge of students' misconceptions about particular mathematics content, recognising students' emerging understanding about a concept).
Pedagogical Content Knowledge (PCK)	Knowledge of content and teaching (KCT)	KCT combines knowing about mathematics and knowing about teaching. KCT understanding and using the pedagogical principles surrounding a particular mathematics concept.
	Knowledge of content and the curriculum (KCC)	KCC involves an appreciation for the resources (e.g., programs, materials) for teaching a mathematics topic at a particular level.

The classification of *specialised content knowledge* (SCK) as a subdomain of SMK rather than PCK is interesting given that SCK is described as being unique to teaching (see Table 2.1) and involves the deconstruction of mathematics in ways that make “particular content visible to and learnable by students” (Ball et al., 2008, p. 400). It may therefore seem curious that SCK is not considered an element of PCK, given its inherent link to pedagogy (Petrou & Goulding, 2011). Irrespective of its position within the structure of the domain map, SCK involves dealing with mathematics in ways that are not typically required, or even appropriate, for settings other than teaching. For example, mathematicians are not routinely required to deconstruct their sophisticated mathematical knowledge into its key components to make it accessible to others.

Some scholars, however, have questioned if it is possible, particularly in practice, to precisely demarcate subject matter knowledge and pedagogical content knowledge in the context of teaching (Marks, 1990; McNamara, 1991; Petrou & Goulding, 2011). Marks argues that because PCK is derived from other types of knowledge “determining where one ends and the other begins is difficult” (1990, p. 8). Indeed, Ball and her colleagues have themselves acknowledged a “boundary problem” with their domain map, recognising that it can be difficult to precisely discern between one knowledge category and another (2008, p. 403). Hurrell (2013) also expresses reservations about the way in which the MKT model is presented, in that it does not imply there are connections between the categories of teacher knowledge.

Other researchers have deliberated the appropriateness of making a clear distinction between subject matter knowledge and PCK given that there are further complexities related to classroom practice (e.g., Marks, 1990; McNamara,1991). For example, some scholars consider Ball and colleagues' domains of teacher knowledge to be limited by the lack of inclusion of teachers' beliefs about mathematics and mathematics teaching (Goulding, Rowland, & Barber, 2002).

Despite these reservations, this MKT model has been influential in paving the way towards identifying and describing the multi-faceted nature of mathematics teacher knowledge. The model has largely been used to develop and implement valid assessment instruments to measure aspects of mathematics teacher knowledge (e.g., Hill et al., 2008;). These instruments have been mainly administered via pen-and-paper test items designed to quantify aspects of teachers' mathematics knowledge for teaching, including their PCK (e.g., Gencturk, 2012; Klauss et al., 2008).

2.3.2 The Chick et al. Framework for Analysing PCK

The Chick and colleagues framework (see Table 2.2) was developed through a combination of theoretical analysis and empirical research. Some elements of PCK within the framework emerged from data collected for the *Knowledge for Teaching Primary Mathematics Project* (KFTPM) which sought to examine teachers' mathematical knowledge enacted in the classroom (e.g., Chick et al., 2006; Chick & Harris, 2007). The majority of the elements however were developed from literature relating to PCK (e.g., Marks, 1990; Shulman, 1987), and teachers' mathematical knowledge more generally (e.g., Askew et al., 1997; Graeber, 1999; Ma, 1999).

Table 2.2

A framework for analysing PCK (Chick, 2007, p. 21)

PCK Category	Evident when the teacher …
Clearly PCK	
Teaching Strategies	Discusses or uses general or specific strategies or approaches for teaching a mathematical concept or skill
Student Thinking	Discusses or addresses student ways of thinking about a concept, or recognises typical levels of understanding
Student Thinking - Misconceptions	Discusses or addresses student misconceptions about a concept
Cognitive Demands of Task	Identifies aspects of the task that affect its complexity
Appropriate and Detailed Representations of Concepts	Describes or demonstrates ways to model or illustrate a concept (can include materials or diagrams)
Explanations	Explains a topic, concept or procedure
Knowledge of Examples	Uses an example that highlights a concept or procedure
Knowledge of Resources	Discusses/uses resources available to support teaching
Curriculum Knowledge	Discusses how topics fit into the curriculum
Purpose of Content Knowledge	Discusses reasons for content being included in the curriculum or how it might be used
Content Knowledge in a Pedagogical Context	
Profound Understanding of Fundamental Mathematics (PUFM)	Exhibits deep and thorough conceptual understanding of identified aspects of mathematics
Deconstructing Content to Key Components	Identifies critical mathematical components within a concept that are fundamental for understanding and applying that concept
Mathematical Structure and Connections	Makes connections between concepts and topics, including interdependence of concepts
Procedural Knowledge	Displays skills for solving mathematical problems (conceptual understanding need not be evident)
Methods of Solution	Demonstrates a method for solving a mathematical problem
Pedagogical Knowledge in a Content Context	
Knowledge of Assessment	Discusses or designs tasks, activities or interactions that assess learning outcomes
Goals for Learning	Describes a goal for students' learning
Getting and Maintaining Student Focus	Discusses or uses strategies for engaging students
Classroom Techniques	Discusses or uses generic classroom practices

The Chick et al. PCK framework is structured around three broad categories: "*Clearly PCK*", "*Content Knowledge in a Pedagogical Context*", and "*Pedagogical Knowledge in a Content Context*". Elements are classified as "*Clearly PCK*" when content and pedagogy are completely intertwined, for example, *knowledge of students' mathematical thinking.* The elements within the "*Content Knowledge in a Pedagogical Context*" category relate to the way mathematical knowledge is held by the teacher, for example identifying critical mathematical components within a concept that are fundamental for understanding and applying that concept (e.g., Baker, 2008; Chick et al., 2006). This category includes Ma's (1999) notion of profound understanding of fundamental mathematics (PUFM). A teacher with PUFM possesses both depth and breadth of understanding of connections within and between mathematical topics and ideas (Ma, 1999).

The third category of the framework, "*Pedagogical Knowledge in a Content Context*" is concerned with generic teacher knowledge applied to particular content, such as the use of strategies to engage students in the learning of a particular mathematical skill or concept (e.g., Chick et al., 2006).

The framework was designed to enable researchers to investigate PCK by applying it to data such as interview transcripts, and actual teaching episodes. Chick and her associates have used the PCK framework to investigate teachers' understanding about aspects of primary mathematics content and their approach to teaching this content (e.g., Baker & Chick, 2006; Chick et al., 2006; Muir & Livy, 2012).

Within the context of secondary mathematics teaching, Vale and her colleagues (Vale & McAndrew, 2008; Vale, McAndrew, & Krishnan, 2010) have used the Chick et al. framework to examine the ways in which a purpose-designed professional learning program can deepen the mathematical knowledge of practicing junior secondary mathematics teachers who lack tertiary mathematics qualifications. The idea of deepening mathematical knowledge is closely tied with making connections between concepts and appreciating mathematical structure (Vale et al., 2010). As such, the "*Content Knowledge in a Pedagogical Context"* category of the Chick and colleagues' framework, which includes *mathematical structure and connections* was of particular relevance to Vale and her colleagues' study. Vale et al. however, point out a lack of specificity relating to the *mathematical structure and connections* component of the framework, given that Chick and her colleagues do not elaborate on the "meaning or significance of mathematical structure" (2010, p. 195).

2.3.3 The Knowledge Quartet

The Knowledge Quartet (KQ) was developed by Rowland and his colleagues during the early 2000s (e.g., Rowland et al., 2005) to explore the ways in which pre-service primary teachers use mathematical content knowledge in their teaching. The research which led to the formation of the KQ was motivated by previous findings highlighting a significant positive association between pre-service teachers' subject matter knowledge (SMK) and their teaching competence (Rowland et al., 2005). As a result, Rowland and his associates sought to investigate the hypothesis that if "superior content knowledge really does make a difference when teaching elementary mathematics, then it ought somehow to be observable in the *practice* of the knowledgeable teacher" (Rowland, 2013, p. 17). In addition, it may be inferred that a teacher with limited content knowledge may misinform his or her students, and/or miss important teaching opportunities (Rowland, 2013).

The Knowledge Quartet consists of four dimensions: foundation, transformation, connection, and contingency. Each dimension comprises several related contributory codes, as shown in Figure 3, which are used to identify and explore the ways in which pre-service teachers' content knowledge and PCK can be observed to "play out" in the classroom (Rowland, 2013, p. 18).

The codes of the KQ were generated through the analysis of video recordings of lessons conducted by 12 pre-service teachers during their final practicum placements (e.g., Rowland et al., 2005). A total of 24 lessons were observed and video-recorded – two lessons from each pre-service teacher – and a grounded theory approach was used to analyse the data (2005). Rowland and his colleagues made a point of ensuring that their analysis of the video-recorded lessons focused on mathematics content knowledge alone, rather than on more general kinds of pedagogical expertise (Rowland, 2008). By way of comparison, such delineation between content and pedagogical knowledge is less evident in the Chick et al. framework which identifies "pedagogical knowledge in a content context" as a broad category of PCK, as discussed in section 2.3.2.

Initially 17 codes were derived from the data based upon lesson episodes and "salient" teaching moments which illuminated aspects of the pre-service teachers' SMK and PCK (Rowland, 2008, p. 281). Following an extended period of rigorous debate and discussion, Rowland and his associates grouped the 17 codes into the four superordinate categories – foundation, transformation, connection, and contingency – which later became known as the Knowledge Quartet. Since the initial conceptualisation of the KQ, Rowland and Turner (2016) explain that ongoing research, in a range of classroom contexts, has led to the inclusion of four additional codes including: *use/misuse of instructional materials* (transformation), *making connections between representations* (connections), *responding to the availability/unavailability of resources* (contingency), and *teacher insight* (contingency).

While Rowland and Turner (2016) express confidence in the basic "anatomy" of the KQ, they also embrace its evolving nature and recognise that the conceptualisation of each dimension is "perpetually open to revision" and refinement (p. 108). A case in point is Petrou's (2009) observation of the different ways in which Year 4 teachers in Cypriot schools used prescribed textbooks in their teaching of mathematics. She noticed that some teachers adapted the content of the textbooks in ways that made it more meaningful and engaging for their students whereas other teachers were unsure how to adapt the content appropriately (Petrou, 2009). Petrou's findings gave rise to the inclusion of the *use/misuse of instructional materials* code to the Knowledge Quartet (Rowland & Turner, 2007). Petrou (2009) suggests that it is not surprising that use/misuse of textbooks, from a transformation perspective, was not addressed in earlier versions of the KQ because the framework was originally applied in the UK context where prescribed textbooks are not commonly used at the elementary level. The addition of *use/misuse of instructional materials* code to the KQ is also highly relevant to the senior secondary mathematics teaching context within which prescribed textbooks are integral to planning and teaching.

Each dimension of the KQ represents a set of comprehensive and higher-order ideas which overarch its constituent codes (e.g., Rowland, 2008). As such, the following sections unpack the four dimensions in relation to both Rowland and colleagues' original conceptualisation of the KQ, and from the perspective of other scholarly literature.

Foundation: Knowledge and understanding of mathematics per se and of mathematic-specific pedagogy, beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics.	• **awareness of purpose** • **adherence to textbook** • **concentration on procedures** • **identifying errors** • **overt display of subject knowledge** • **theoretical underpinning of pedagogy** • **use of mathematical terminology**
Transformation: The presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations	• **choice of example** • **choice of representation** • **use of instructional materials** • **teacher demonstration (to explain a procedure)**
Connection: The sequencing of materials for instruction, and an awareness of the relative cognitive demands of different topics and tasks	• **anticipation of complexity** • **decisions about sequencing** • **recognition of conceptual appropriateness** • **making connections between procedures** • **making connections between concepts**
Contingency: The ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events	• **deviation from agenda** • **responding to students' ideas** • **teacher insight during instruction** • **responding to the (un)availability of tools and resources**

Figure 2.3. The Knowledge Quartet: Dimensions and Contributory codes (Rowland, 2013)

Foundation

The Foundation dimension is underpinned by the assumption that teachers' mathematical *knowledge* and *beliefs* about mathematics have the potential to inform and influence their instructional decisions in fundamental ways (Rowland et al., 2005). Within the context of the *foundation* dimension, mathematical *knowledge* refers to teachers' knowledge of mathematics content, whereas *beliefs* include: teachers' beliefs about the nature of mathematics, the purpose of learning mathematics, and beliefs about how mathematics should be learnt and taught (Rowland et al., 2005). Rowland and his colleagues acknowledge an alignment between the *foundation* dimension of the KQ and the *comprehension* phase of Shulman and colleagues' cycle of *pedagogical reasoning and action* (2005). That is, mathematics teachers need to understand the substantive and syntactic knowledge structure of the content they are required to teach.

The beliefs aspects of the *foundation* dimension include: beliefs about the nature of mathematics, beliefs about the purposes of learning mathematics and specific topics, and beliefs about preferred ways of teaching and learning mathematics (Rowland et al., 2005). Rowland and his colleagues highlight the work of scholars such as Hersh (1997) who emphasises the inevitable influence that teachers' beliefs about the nature of mathematics have on their instruction: "the student takes in the teacher's philosophy through her ears and the text book's philosophy through her eyes" (Hersh, 1997, p. 264). The ways in which Ernest (e.g., 1989) distinguishes among different views about the nature of mathematics as a discipline provides a useful framework upon which inferences may be drawn about people's mathematics-related beliefs. Ernest identifies three broadly different views of mathematics: the problem solving view where mathematics is seen as a continually evolving field of human enquiry, the Platonist view where mathematics is viewed as a “static but unified” body of knowledge that exists independently of human endeavour, and the instrumentalist view where mathematics is believed to be a set of “useful” rules, facts and skills (1989, p. 20).

While the *foundation* dimension is concerned with the knowledge and beliefs held by the teacher independently of his/her work in the classroom, the remaining three dimensions of the KQ are concerned with the ways in which teacher knowledge is enacted both in the planning and implementation of lessons (Rowland et al., 2005).

Transformation

The *transformation* dimension of the KQ in similar in nature to the *transformation* stage of Shulman’s cycle of *pedagogical reasoning and action* during which teachers transform their personal understanding of specific content knowledge in pedagogically powerful ways for their students (Rowland et al., 2005).

The *transformation* dimension “concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself” (Rowland et al., 2005, p. 261). Drawing on the work of Shulman, Rowland and his colleagues highlight the importance of teachers’ choice of representations (e.g., analogies, diagrams, explanations, teacher demonstrations, and illustrations) to convey ideas to students. Similarly, teachers’ use of instructional materials, including textbooks, involves transforming the content for teaching. For example, critiquing textbooks involves noticing errors, omissions, or ambiguities in the presentation of content, and adapting content into forms more suited for learning (Petrou, 2009).

Teachers’ choice and use of examples is a key element of the KQ, and also feature in other frameworks (Chick et al., 2006). The use of exemplification in mathematics teaching has been widely researched (e.g., Bills et al., 2006; Watson & Mason, 2006). An "instructional example", according to Zodik and Zaslavsky (2008), is any example considered within the context of learning a particular topic. Good instructional examples invite students to build appropriate generalisations about a mathematical idea by directing them towards key features that make it exemplary. Examples are frequently utilised in teaching senior secondary mathematics.

Rowland and his colleagues (2009) make the broad distinction between two common uses of examples in mathematics teaching: to introduce mathematical concepts and procedures, and in the provision of practice exercises to consolidate learning. The former typically includes a teacher demonstration or explanation which draws attention to key features, and possible difficulties or pitfalls, relating to the new concept or process (Rowland et al., 2009). The latter use of examples focuses on skills development through the completion of exercises after the new concept or procedure has been introduced (2009).

Rowland and his associates (2009) point out that examples used to introduce and consolidate new concepts or procedures, tend to focus on substantive mathematical knowledge (knowledge *of* mathematics); whereas examples involving syntactic knowledge (knowledge *about* mathematics) are less prevalent, particularly in the elementary classroom. Examples involving syntactic knowledge, such as general mathematical structure and reasoning, often represent particular cases from a larger class from which generalisations can be made (Zodik & Zaslavsky, 2008). In fact, often it is not the specific example or even the answer that is the most important, but the general principle illuminated by the example. Teachers' choice and use of examples therefore have important pedagogical implications and can also serve to address students' misconceptions about mathematical ideas. Zazkis and Chernoff

(2008) highlight the role of the counterexamples as a powerful way of encouraging students to confront and work through their misconceptions. A counterexample is a particular case of a general claim that refutes that claim.

Rowland and his colleages' observations of some of the ineffective ways in which novice elementary teachers use examples in teaching (e.g., randomly generating examples, or inadvertently obscuring the role of key variables) serves to illuminate the high level of expertise required in choosing pedagocially powerful examples (Rowland et al., 2009). It is widely accepted that teachers' capacity to select or devise appropriate examples in mathematics is linked to the depth and quality of their mathematical content knowledge (e.g., Leikin & Levav-Waynberg, 2007).

Within the *transformation* dimension, Rowland et al. (2009) also include the effective use of teacher questioning. Teacher questioning is a critical and challenging part of a teacher's work (Boaler & Brodie, 2004). Drawing on the work of Shulman (1987), Boaler and Brodie also emphasise that good questioning is both cognitively challenging and requires well-developed PCK (2004). One of the findings from Boaler and Brodie's study into mathematics teacher questioning indicates that the most common types of teacher questions involve information gathering (e.g., "what is the value of x in this equation?"), or leading students through a method of solution (e.g., "How would you plot that point") (2004, p. 777). By contrast, there was far less focus on questions designed to extend students' thinking such as "would this work with other numbers?" (p. 777).

Similarly, Mason (1998) suggests that the tendency for teacher questioning to become habitual or limited in scope relates to the "didactic tension" that results from the perception that "the more clearly the teacher indicates the behaviour sought, the easier it is for students to display that behaviour without generating it from understanding" (p. 1). For example, Mason describes Bauersfeld's (1994) notion of the *funnelling effect* whereby a teacher "sees something and tries to get the students to see it" by asking a series of funnellingand increasingly directed questions (p. 14). Mason suggests that while funnelling questions can be effective, if they are overused or become habitual then this can be limiting in terms of enhancing students' thinking and learning (1998). Therefore, Mason advocates for teachers to enhance their questioning practice by seeking to pose questions that reflect their own sense of mathematical structure, thereby supporting students to make sense of mathematics within the context of genuine enquiry (1998).

Connection

The *connection* dimension of the KQ is underpinned by the notion of coherence: coherence in relation to teaching and learning, and coherence as a feature of the discipline of mathematics itself (Rowland et al., 2005). Rowland and his colleagues highlight the influence of the work of Askew et al. (1997), Ball (1990), and Ma (1999) who foreground the crucial role of connected knowledge in teaching and learning mathematics.

While the *connection* dimension delineates between *connections between concepts* and *connections between procedures*, it is imbued with the idea that concepts and procedures are inherently linked (Rowland et al., 2009). Drawing on the work of Gray and Tall (e.g., 1994), Rowland and his colleagues (2009) suggest that learning concepts often grows from familiarity with procedures. Gray and Tall (1994) used the term "procept" to describe the combination of an abstract mathematical concept and the procedure associated with it. Rowland et al. (2009) highlight the *mean* as an example of a "procept" because it is difficult to separate the concept of the mean as a measure of centre of a data set, from the process of obtaining the mean (i.e., summing the numbers in the data set and dividing by the number of elements in the set). In other words, "exemplifying a procedure frequently paves the way for the acquisition of a concept, and that the distinction between procedure and concept is not clear-cut" (Rowland et al., 2007, p. 71).

Star (2005) also raises interesting issues in relation to the distinction between procedure and concept by highlighting a general tendency to oversimplify knowledge types in terms of their quality. That is, Star warns against the tendency to adopt a simplistic view that conceptual knowledge is inherently deep (richly connected) and procedural knowledge is inherently superficial (without rich connections). He therefore advocates for a reconceptualization of conceptual and procedural knowledge to acknowledge that *both* conceptual and procedural knowledge can be either superficial or deep. According to Star, flexibility is a key attribute of deep procedural knowledge (2005). Flexibility relates to the capacity to notice the most efficient strategy for solving a particular mathematical problem. It is however important to acknowledge that deciding upon the most efficient strategy is, itself, nuanced (Star, 2005). For example, "is the most efficient strategy the one that is the quickest or easiest to do, the one with the fewest steps, the one that avoids the use of fractions, or the one that the solver likes the best?" (Star, 2005, p. 409). A person with superficial procedural knowledge may depend upon a standard technique or algorithm to solve a particular problem, possibly resulting in a less efficient solution process, and limited capacity to solve unfamiliar problems (2005). On the other hand, someone with deep procedural knowledge is likely to "navigate his or her way through this procedural domain" using methods which are not confined to "overpracticed" or rote learnt techniques (Star, 2005, p. 409).

In their response to Star's proposal for a reconceptualization of procedural knowledge, Baroody, Feil, and Johnson (2007) maintain that deep procedural knowledge cannot exist independently of conceptual knowledge. In other words, deep procedural knowledge, which involves understanding how the steps in a mathematical procedure are interrelated, cannot exist without understanding the conceptual basis of each of these steps (Baroody et al., 2007).

Within the *connection* dimension, Rowland and his colleagues also include the sequencing of topics of instruction, including the ordering of examples. While the latter could be considered an aspect of *choice of examples* from the *transformation* dimension, sequencing of content is logically placed in the *connection* dimension because it relates to the notion of coherence. Rowland and his colleagues emphasise that "deliberations and choices" about the sequencing of content relate to connections within the mathematics content itself, as well as "an awareness of the relative cognitive demands of different topics and tasks" (Rowland et al., 2009, p. 31). Hence *anticipation of complexity* is a key aspect of the connections dimension. The ways in which teachers manage the cognitive demand of mathematics tasks is also important. For example, some scholars (e.g., Anthony, 1996; Henningsen & Stein, 1997) suggest that when teachers routinely and systematically reduce the cognitive demand of mathematical tasks, the depth of independent thinking required by the students is, in turn, reduced.

Contingency

The *contingency* dimension is concerned with the ways in which teachers respond to classroom events as they unfold, and which are "almost impossible to plan for" (e.g., Rowland et al., 2005). A key component of *contingency* includes teachers' readiness to *respond to pupil ideas* and hence their preparedness to *deviate from the* *lesson agenda.* These components are underpinned by a constructivist theory of learning where students' mathematical ideas reflect "*their* knowledge construction" which may or may not align with what the teacher intended them to learn (Rowland et al., p. 263). Rowland and his associates highlight that students' ideas (e.g., unsolicited questions or comments) are indicators of their meaning-making (2005). The ways in which teachers engage (or not) with students' unexpected ideas, have important implications for the meaning-making process. For example, ignoring a student's response, setting it aside, or writing it off as "wrong" (without further discussion), can be interpreted as a lack of interest in what the student has "come to know as a consequence, in part, of the teacher's teaching" (Rowland et al., 2005). Rowland, Thwaites, and Jared (2015) identify three ways in which teachers handle unexpected responses from students: ignoring the response, acknowledging the response but putting it aside, and both acknowledging and "incorporating the response" (p. 79). In other words, these three kinds of teacher responses reflect varying degrees of readiness to *deviate from the lesson agenda*, another key component of the *contingency* dimension.

Before pursuing the idea of contingency any further, it is important to highlight that a knowledgeable, and likely experienced, teacher is able to anticipate, to some extent, the events of a lesson based on his or her knowledge of factors such as: students' common errors and misconceptions, an awareness of content students find easy or difficult, and knowledge of how students respond to particular "instructional stimuli" (p. 77).

Nevertheless, unpredictable moments do arise in classrooms (Rowland et al., 2015). Schoenfeld (1998), as cited by Rowland and colleagues (2015), unpacks the complexity of teachers' in-class decision-making. Schoenfeld (1998) describes how triggers of contingency such as an unexpected question or comment from a student can impact upon a teacher's lesson agenda. For example, if a comment from a student reveals a misunderstanding about a mathematical idea, the teacher must decide on how important it is to raise the misunderstanding with the rest of the class (1998).

Rowland and his associates identify two other components of the *contingency* dimension which are not directly initiated by students. These components include *teacher insight* and *responding to the availability (unavailability) of tools or resources*. The former relates to classroom situations where the teacher chooses to stop and reflect-in-action and change tack (e.g., change an example or representation). The latter involves responding to the presence or sudden unavailability of an intended teaching resource (e.g., a particular software program failing to load). Teachers often use artefacts – including digital technologies – to "mediate abstract, intangible concepts" to help students make sense of them (Rowland, et al., 2015).

Each element of the contingency dimension requires the teacher to call upon specific knowledge in the moment and knowing-to-act (Mason & Spence, 1999). The idea of knowing-to-act involves more than possessing a bank of knowledge about effective teaching, rather, knowing-to-act calls upon knowledge in the moment it is needed. For example, it is "one thing to notice an absence of something from a learner, but quite another thing to have a sensible pedagogical action come to mind when needed" (Mason & Davis, 2013, p. 183). In addition, the authors suggest that teachers' moment-by-moment pedagogical choices of action are potentially the most influential source of insight into mathematics teacher knowledge. Similarly, Mason (2008a) associates a "richly conceived" and nuanced PCK with "being mathematical with and in front of the learner" (p. 307). The idea of "being mathematical with and in front of the learner" as developed by Mason, involves "sensitivity" to noticing opportunities to initiate actions in the classroom (2008, p. 20). In this sense, the idea is associated with awareness on the part of both the teacher and the students. That is, the teacher needs to "be aware of what the learners are not yet aware of" (p.20). As such, being mathematical with and in front to the learner involves choosing actions that direct learners' attention in ways that develop their awareness of salient aspects of the mathematics task at hand.

Mason and Davis (2013) assert that the most critical factor in teachers' capacities to engage productively with their students in moment-by-moment classroom interactions, is their own mathematical thinking. This kind of mathematical thinking includes knowledge of mathematics content as well as the "scope and range of mathematical thinking, associated pedagogical strategies, and didactic tactics, that are available to come-to-mind, in the moment" (p. 184).

The Application of the Knowledge Quartet

The KQ has mostly been used to explore the ways in which pre-service primary teachers' mathematics-related teacher knowledge plays out in the classroom (e.g., Petrou, 2009; Rowland et al., 2005). Some studies have used the KQ in the development of pre-service primary teachers' mathematics-related teaching as part of their undergraduate university courses (e.g., Liston, 2015; Livy, 2010). Relatively few studies, however, have applied the KQ within the secondary mathematics teaching context, or beyond. In addition, the framework has been recently applied to the work of Mathematics Teacher Educators (e.g., Chick & Beswick, 2017; Muir, Wells, & Chick, 2017).

Most notably Rowland and his colleagues (e.g., Thwaites, Jared, & Rowland, 2011) have themselves tested the potential of the KQ as an analytical tool within the secondary mathematics teaching context. While Rowland and his associates deemed the framework to be potentially useful for analysing teaching in the secondary context, they did allude to the possibility that additional codes (particularly in the transformation dimension) may be needed to capture the nature of more extended mathematical explanations (Thwaites et al., 2011). These authors suggested that mathematics content studied at the secondary level is more complex than in the primary level and linked this notion to Potari and her colleagues' (2007) claim that the integration of content and pedagogy is more difficult to achieve at the senior secondary level.

2.4 Research on Teacher Knowledge at the Senior Secondary Mathematics Level

Research into senior secondary mathematics teaching and learning has mainly focused on specific areas such as the use of technology (e.g., Geiger, Faragher, & Goos, 2010; Kendal & Stacey, 2001). A few other studies have alluded to the complex relationship between different aspects of teacher knowledge at the secondary, and senior secondary, level (e.g., Brown, 2002; Leikin & Levav-Waynberg, 2007; Potari et al., 2007).

The use of CAS in the senior secondary classroom offered new approaches for teaching and learning mathematics at this level. For example, Geiger and his colleagues (2010) explored the potential of CAS technologies to enhance students' understanding of mathematical modelling. The authors describe a teaching and learning episode, in their study, where a group of students used their CAS calculators to model the decline in population of a specific animal to extinction and were surprised when the answer displayed "false". The teacher strategically intervened by assuring the students that no syntax error had been made, and they should think about the assumptions they had made. The students came to realise that it is not mathematically valid to equate an exponential function to zero, despite their assumption that extinction meant a population of zero. The authors used this lesson episode to illustrate the complex and multi-faceted nature of teacher knowledge at the senior secondary level. The teacher's capacity to recognise and act upon teachable moments, such as the one described above, requires "the disposition, mathematical expertise, technological competence and confidence to explore and promote students' mathematical knowledge and their understanding of mathematics in context" (p. 64).

The complexity of teacher knowledge at the secondary and senior secondary levels was also explored by Potari and her colleagues (2007). Their study focused on secondary teachers' mathematics knowledge for teaching calculus and observed the high level of sophistication required by the teacher to integrate content and pedagogy. They associated this idea with the development of the teachers' PCK. In addition, these authors postulated that mathematics teacher knowledge is topic-specific, after observing the depth with which individual teachers approached their teaching of calculus compared to other topics, such as geometry. As such, Potari and her associates suggested the need for more research into ways in which existing teacher knowledge frameworks can be used to explore secondary and senior secondary teachers' knowledge across different mathematics topics.

Other key research into the nature of teacher knowledge at the senior secondary level includes Leikin and Levav-Waynberg's (2007) examination of the influence of the broader educational context on the teachers' preferred pedagogical approaches. The authors observed a general reluctance by their participating teachers, to embrace the use of alternative teaching approaches that were designed to assist students to build rich and meaningful mathematical connections. These findings were attributed to the situated nature (Lave, 1996) of teachers' practice, where instructional decisions were influenced by factors including school and community expectations, curricular requirements, and assessment. The teachers tended to adhere to well-established classroom norms where textbook problems were viewed as "vehicles" for teaching or practicing specific techniques and therefore providing students with "secure tools" with which to meet curriculum requirements (2007, p. 366). Leikin and Levav-Waynberg highlight a discrepancy between research-based recommendations about the value of rich, conceptually focused mathematics teaching and the reality of classroom teaching. They attribute this discrepancy to a lack of common beliefs and norms among stakeholders within the wider education context (e.g., researchers, teachers, and educational authorities) and suggest that any reforms to teaching approaches would need to stem from the curriculum level.

Similarly, Brown's (2002) study into the characteristics of teaching and learning approaches at the senior secondary mathematics level supports a similar view that any proposed reforms must originate from the broader curriculum level to be sustainable. The findings of Brown's study indicate that both teachers and students tended to use surface strategies and "achievement-oriented" goals (p. 7). In addition, his study provided evidence that teachers often attribute obstacles related to course constraints (e.g., external examinations) to their perception of the ways in which senior secondary mathematics courses should be taught and learnt. In this sense Brown posed the idea that teachers can be "victims" of the pre-tertiary qualification system (p. 7).

2.4.1 Summary of Research and Implications for Present Study

The previous section discussed several studies which have focused on aspects of senior secondary mathematics teacher knowledge. For example, Geiger and his colleagues (2010) drew attention to the complexity of teacher knowledge at the senior secondary level by illustrating the way in which a teacher simultaneously drew upon content knowledge, technological competence, and insight into students' thinking. Potari and her associates (2007) alluded to complexities that may be unique to teaching advanced mathematics content, given the level of sophistication required to integrate content and pedagogy at this level. In addition, the authors highlighted the potential for further research into teacher knowledge at the senior secondary level using existing theoretical frameworks.

Leikin and Levav-Waynberg's (2007) study foregrounded the influence of broader institutional constraints on the enactment of senior secondary mathematics teachers' knowledge. Their findings indicated that teachers tend to adhere to standard approaches to solving mathematics problems, in order to provide students with the secure learning environment they believed to be conducive to achieving success in final examinations.

Few studies, however, have explored the enactment of senior secondary mathematics teachers' knowledge in a detailed and multi-faceted way. In addition, the application of theoretical frameworks designed to unravel different aspects of this knowledge and the interactions between them, has been largely unexplored.

2.5 Summary of the Chapter

This chapter provided an overview of the multi-faceted nature of teacher knowledge, followed by a discussion of the evolution of understanding of PCK as a complex and nuanced aspect of this knowledge. A description of key mathematics teacher knowledge frameworks, the ways in which they have been used to examine teaching, and their potential for use in the senior secondary mathematics context was also addressed. The chapter concluded with a focus on key research in the context of the senior secondary mathematics classroom, highlighting insights and the ways in which these have informed the current study. The next chapter addresses the qualitative research methods used in this exploratory study into PCK in the senior secondary mathematics classroom.

Chapter 3

Methodology

3.1 Introduction

This chapter presents the design and implementation of this study, beginning with the overarching research perspective, followed by the ethical considerations and procedures employed in the generation of the data from multiple sources. Details of the ways in which these data were analysed are discussed, followed by a discussion of issues concerning the trustworthiness of the study.

Qualitative research methods were used to explore evidence of PCK in the interactions between teachers and their students in the senior secondary mathematics classroom. This exploratory study was guided by the following research questions:

1. What aspects of mathematical pedagogical content knowledge are evident in the interactions between teachers and their students during the teaching and learning of senior secondary mathematics content?
2. What aspects of mathematical pedagogical content knowledge do teachers discuss and attribute their instructional decisions to when analysing their interactions with students during the teaching and learning of senior secondary mathematics content?

3. What aspects of mathematical pedagogical content knowledge are identified by students as having an impact on their learning of senior secondary mathematics content?

3.2 Research Perspective

3.2.1 Qualitative Research

The qualitative research tradition is overarched by a broadly interpretivist philosophical position that is concerned with ways in which the social world is interpreted or experienced (Denzin & Lincoln, 2005). Qualitative researchers “study things in their natural settings, attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them” (p. 3). For this study, a qualitative research paradigm was appropriate because PCK is a phenomenon given that it can be observed to play out in the teachers’ interactions with their students in the classroom. PCK itself may be broadly defined as the intricate blend of content and pedagogical knowledge that constitutes all that is needed to teach a mathematics topic or idea.

According to Mason (2002), qualitative research techniques are rich and varied; however, many share fundamentally similar attributes. These include methods of data generation that are “flexible and sensitive to the social context in which data are produced” and methods of analysis that “involve understanding complexity, detail, and context” (p. 3). This study examines PCK in action in the senior secondary mathematics classroom through multiple sources of data including lesson observations, video footage of lessons, post-lesson interviews with teachers and students, and short written reflections from students.

3.3 The Study Design

A case study research design was chosen for this project because, as described by Stake (1995), it enables the researcher to capture the complexity of a phenomenon and come to understand its interaction with the environment within which it is situated. Similarly, Yin broadly describes qualitative case study research as conducting an empirical investigation of a "contemporary phenomenon in its real-life context" using multiple sources of evidence (1981, p. 98).

Stake (1995) categorises case studies as being one of intrinsic, instrumental, or collective. Of most relevance to this study is the collective case study, because it involves the exploration of issues from multiple instrumental case studies. Instrumental case studies are concerned with the phenomenon under research rather than the identifying the characteristics of a particular case, which applies to intrinsic case studies. The case itself is a "bounded system" of interest (p. 2). The concept of a bounded system is fundamental because it gives "great prominence to what is and is not 'the case'" (Stake, 1978, p. 7).

The case in this research project is the PCK enacted in the classroom, and/or discussed by the participating teachers and their students. PCK, for the purposes of this study, encompasses, but goes beyond, Shulman's original conceptualisation of the construct by also including teachers' knowledge of mathematical content, curriculum knowledge (e.g., use of textbooks and other instructional resources), knowledge of assessment, as well as teachers' conceptions of the ways in which mathematics is best taught and learnt. This interpretation of PCK is consistent with the broad and inclusive views of PCK supported by scholars including Magnusson et al. (1999) and Hashweh (2005) as discussed in Section 2.2.3.

In keeping with the case study research paradigm, data sources in this study were: lesson observations and video footage of lessons, post-lesson interviews with teachers and students, and short written reflections from students. These data sources are discussed further in Section 3.7. An outline of how the research questions relate to the research approaches, data generation techniques, and analysis of the senior secondary teachers' PCK is presented in Table 3.1.

Table 3.1

Research Design

Research question	**Data generation**
RQ 1	Lesson observation and video footage of lessons
	Written summaries of video recordings of lessons
	Post-lesson interviews with each teacher.
	Full transcriptions of post-lesson teacher interviews
	Full transcriptions of post-lesson student focus-group interviews
RQ 2	Post-lesson interviews with each teacher.
	Full transcriptions of post-lesson teacher interviews
RQ 3	Full transcriptions of post-lesson student focus-group interviews and students' short-written reflections

3.4 Case Selection

The three senior secondary mathematics classes in this study were from two schools in Northern Tasmania, identified hereafter as School A and School B. Two of the classes were from School A, a non-government, co-educational, inner-regional secondary school (Year level range 7-12), with an enrolment of approximately 1400 students. The third class was from School B, a non-government, co-educational, combined school (Year level range prep-12), with a student enrolment of approximately 600. The Index of Community Socio-Educational Advantage (ICSEA) for School's A and B were 1020 and 1050 respectively (data retrieved from https://www.myschool.edu.au/ in 2015), at the time the research was conducted.

The researcher selected the schools based on the availability of teachers of senior secondary mathematics, and convenience of location to the researcher. The participant classes from School A included one class of 18 students and their teacher (Mr Jones), and another consisting of 11 students and their teacher (Mr Taylor). The class from School B consisted of nine students and their teacher (Mr McLaren). Details about the teacher and student participants are given in Section 3.6. All names are pseudonyms.

The context within which PCK was explored in this study was Mathematics Methods (course code MTM315). Mathematics Methods was a pre-tertiary mathematics course accredited by the Tasmanian Qualifications Authority (TQA) now called the Office of Tasmanian Assessment, Standards and Certification (TASC). This course was one of three pre-tertiary Year 11/12 mathematics courses offered in Tasmanian schools and colleges at the time the study was conducted. Mathematics Methods was selected because it focused on topics such as calculus and statistics, with greater rigor than the applied mathematics course (Mathematics Applied MTA315) and therefore offered a richer context within which to explore senior secondary mathematics teachers' PCK. In addition, Mathematics Methods attracted a greater number and wider variety of student enrolments than the most demanding of the three pre-tertiary mathematics courses, Mathematics Specialized (MTS315), for which MTM315 was a highly recommended precursor. The versions of the mathematics courses highlighted above were current at the time the research was conducted but have since been updated – although the core content of the courses remains fundamentally similar.

Mathematics Methods was designed for students whose future career pathways involve mathematics, such as engineering, the sciences, technology, commerce and economics, and health and social sciences (Tasmanian Qualifications Authority, 2014). The key topics studied in the course include: functions and graphs, circular functions, calculus, statistics and probability.

The lessons observed in this study involved either calculus, or statistics and probability. These topics were selected because the classes were working on calculus followed by probability and statistics, at the time of year the lesson observations took place. In addition, it was useful to explore the ways in which two different topics were taught.

3.5 Ethical Considerations

3.5.1 Seeking Permission from Schools

Ethical permission to conduct this study was sought from the Human Ethics Research Committee (HERC) Tasmania Network in February 2014 and granted in April 2014 (see Appendix 1). Following approval from the HERC, the researcher requested permission, via email from the Tasmanian Catholic Education Office to conduct her research in a Catholic School in Northern Tasmania. Permission was granted in May 2014 (see Appendix 2).

In May 2015 approval to conduct the study was also sought from a second school (School B) from a different sector in Northern Tasmania for which principal approval was the highest level of permission required. Approval was granted in May 2015 via the school's assistant principal. It is worth noting that approval from the Tasmanian Education Department was also sought and granted in May 2015, but the teacher who subsequently agreed to participate had to withdraw before the study commenced due to ill-health.

In the interests of attempting to achieve a gender balance with the teacher participants, two female teachers of Mathematics Methods MTM315 from a Tasmanian Education Department College were invited to participate in the study, but both declined. There were no female teachers of Mathematics Methods MTM315 at either School A or School B at the time of invitation.

3.5.2 Obtaining Participants' Consent

After obtaining approval from the Tasmanian Catholic Education Office, permission was sought, via email, from the principal of School A to invite the Mathematics Methods teachers from his school to participate in the study. The researcher then invited teachers to participate based on their years of experience teaching Mathematics Methods – the teacher with the most experience was invited first and then the second most experienced teacher was invited so that the maximum number of teacher participants for each school was two. This selection criterion was based upon the assumption that experienced teachers will have had time to develop and articulate their PCK for teaching senior secondary mathematics. The process for inviting teachers to participate in the study was repeated for School B, in the following year.

Based on the selection processes described above the researcher invited two teachers from School A, and the only teacher of Mathematics Methods (MTM315) from School B, to participate in the study. The invitations were sent via email (see Appendix 3) with the teacher information sheet and consent forms (Appendix 4 and 5 respectively) were sent as attachments.

After the teachers had agreed to participate in the study, the students in their classes were invited to take part. Given that many of the students were 17 years of age it was necessary to seek informed consent from their parents/caregivers. Out of courtesy and respect, informed consent was also sought from the students themselves. The researcher visited each of the three classes, taught by the participating teachers, towards the end of one of their Mathematics Methods MTM315 lessons to explain the nature of the study and to distribute the parent/caregiver and student information letters (Appendix 6 and 7 respectively) and consent forms (Appendix 8 and 9 respectively). In addition, the researcher gave a verbal explanation, to the students in attendance, of the contents of the information sheets and consent forms.

The completed consent forms were returned to the researcher in sealed envelopes via the students' Mathematics Methods teacher prior to commencement of the study.

3.5.3 Non-Intrusive Data Generation Techniques

The researcher appreciated the high stakes involved in studying a pre-tertiary subject such as Mathematics Methods and therefore set out to ensure that data generation was as unobtrusive as possible. Lesson observations, including video-recording aspects of the lessons, took place while participants were engaged in the usual activities of teaching and learning. It was anticipated that students who did not consent to be involved in any aspects of the study would still attend the lessons but would not be video-recorded or interviewed. In addition, the researcher would not record or photograph any aspects of their involvement in the lesson observations. As it transpired, however, all students in each of the three classes agreed to participate in one or more aspects of the study (e.g., participate in focus-groups, complete short-reflections) and all students consented to having their participation in the lessons observed and video-recorded. During each lesson observation (six in total per class) the researcher positioned herself at a desk within clear view of the whiteboard and used a hand-held digital camera to video-record aspects of the lesson (Section 3.7.1 details the way in which video was used in the lessons). Care was taken to ensure the video-recording was as unobtrusive as possible including sounds switched to silent on the device.

It was also important to ensure that both teacher and students felt comfortable with their own and each other's participation in the study. All students within each class consented to participate in at least one aspect of the research. All interview questions and the student short-reflection questions focused on the specific actions carried out by the teacher to assist students in their learning. The teacher participants were experienced, well regarded by their school communities, and were not likely to feel concerned about what might be discussed in the focus groups with students.

As Mathematics Methods (MTM315) was an externally assessed pre-tertiary course no data were generated during the three-week period leading up to the midyear examination, or during the final school term when participants were engaging in intensive revision and preparation for the final examination.

3.5.4 Data Storage, Reporting, Obligations, Privacy and Anonymity

Pseudonyms were used in all transcripts, lesson summaries, and in subsequent reporting of the study. Audio files were stored in password protected digital audio files on a secure server at the University of Tasmania, Launceston Campus.

Care was taken not to include any identifying information about the participating teachers in notes made by the researcher during observations of lessons or during the interviews with teachers. Files containing transcripts of the students' focus-group interview responses were labelled using pseudonyms, as were the photographs of participants' work and their written responses to the short-reflection questions. A file linking participants' names to pseudonyms was stored on a password-protected computer, separate from the transcripts of the interview, and only the researcher had access to this material. The student short-written reflection data were individually identifiable to enable the researcher to refer to a student's own response to the post lesson questionnaire during their focus group interview.

3.6 Participants

3.6.1 The Teacher Participants

The three teacher participants were Mr Jones and Mr Taylor, both from School A, and Mr McLaren from School B. Mr Jones had 27 years of experience teaching secondary mathematics and science, including nine years teaching Mathematics Methods MTM315 and previous versions of the course. Mr Taylor had been teaching secondary mathematics for over 40 years and had taught MTM315 and equivalent courses for 15 years. Mr McLaren had taught secondary mathematics and science at School B for 15 years with 11 years of experience teaching pre-tertiary Mathematics

Methods including MTM315. All three teachers had studied mathematics at the tertiary level.

3.6.2 The Student Participants

The students in this study were the Year 11 or 12 students who were in one of the three Mathematics Methods (MTM315) classes taught by Mr Jones, Mr Taylor, or Mr McLaren. There were 18 students (8 female, 10 male) in Mr Jones' class, 11 students (3 female, 8 male) in Mr Taylor's, and nine students (2 female, 7 male) in Mr McLaren's class. All students in each class consented to participating in one or more aspects of the study.

3.7 Data Generation Methods

The methods used in this study included lesson observation, video-recording of the lessons, interview, and written responses. This is consistent with a key finding from Depaepe, Verschaffel, and Kelchtermans' (2013) review into the ways in which PCK has pervaded mathematics education research. Depaepe et al. (2013) found that studies which seek to capture teachers' PCK in the act of teaching typically use several data sources including observation and video-recording of the lessons, interviews, and written documentation.

The following subsections discuss each method of data generation including a theoretical justification for the choice of method, and a description of the instruments used where applicable. It is important to highlight the complex and inevitable role of the researcher as an instrument of data generation and analysis in qualitative research (Mason, 2002). The researcher in this study was aware that her own mathematical knowledge influenced how she observed and interpreted the teaching and learning interactions and how this influenced the questions she chose to ask the teachers and students in their post-lesson interviews.

3.7.1 Observation and Video-Recording of Lessons

Lesson observation, as a method of data generation, was used in this study because PCK is a dynamic construct encompassing teacher knowledge in action. It therefore makes sense to investigate PCK in the context within which it is enacted – in this case, the senior secondary mathematics classroom. According to Mason (2002) observation allows the "generation of multidimensional data on social interaction in specific contexts as it occurs" (p. 85). The classroom observations focused on evidence of PCK in the teaching and learning interactions between the teacher and participating students. Of particular interest were the instructional phases of each lesson, during which the teacher explained mathematical content to the whole class and demonstrated worked solutions to examples. Field notes documenting instances such as the teacher's use of a particular example or representation were generated by the researcher as she observed PCK in action. The main purpose of the field notes was to develop questions to include in the post-lesson teacher interview and/or the student focus-group interview. For example, an interview question derived from the field notes may probe for further information in relation to a teacher's instructional decision from his perspective.

The use of video to generate data in this study was necessary because the researcher was unlikely to recognise and recall as many interesting teaching and learning incidents in detail by using direct observation alone. Mousley (1998) highlights that video-recording in mathematics education research provides permanent visual and reviewable documentation of lesson events that may be later described in detail. Moreover, given the complexity of the mathematics content involved in this study, the researcher needed to revisit the video footage repeatedly, particularly when making connections between corresponding teacher and student data. The role of the researcher in any classroom video study is described by Clarke (2013) as a "purposeful act by the researcher to selectively construct a data set", aligned with the type of data analysis anticipated and the specific research questions under study (Clarke, 2013, p. 227).

In this study, the researcher chose to video-record the instructional phases of each lesson for the purposes of capturing, in detail, multiple aspects of a teacher's PCK in action such as choice of examples, questioning, the use of representations, and responding to student thinking. The instructional phases of the lessons included those phases during which the teacher provided whole-class explanations and or worked solutions to examples. In the interest of remaining as unobtrusive as possible and given the limitations of the use of only one camera, the researcher chose not to film other aspects of the lessons, including the students' seatwork and small group conversations among students and or their teacher.

3.7.2 Interviews

Semi-structured interviewing was used in this study because it involves the active exchange of dialogue between the researcher and interviewee, rather than a one-way question and answer format (Holstein & Gubrium, 1995). The purpose of this exchange of dialogue is to allow both researcher and interviewee to bring ideas to the fore so that meaning is actively constructed in a continually unfolding process (Holstein & Gubrium, 1995).

The post-lesson teacher interviews were designed to generate data relating to PCK from the teachers' perspectives. Calderhead (1996) asserts that teachers' own perceptions, judgments, and reflections are crucial for any meaningful exploration of teaching in action given the complexity and nuance of their instructional decisions. In addition, teachers' justification for their own instructional choices is a significant component of PCK, and classroom observation alone is not sufficient in unravelling PCK (1996). The interviews were semi-structured in nature and involved a combination of pre-prepared questions as well as questions arising from the day's lesson observations. The pre-prepared questions (see interview schedule in Figure 3.1.) were designed to elicit responses related to the teachers' planning for teaching particular mathematics content, and their knowledge of the common difficulties that students often experience with this content. The interview questions that arose from the lesson observations, however, focused on observations of specific teaching and learning interactions that occurred during the day's lesson, as well as general questions about the teacher's own identification and perception of these interactions.

Following are examples of the generic questions that teachers were invited to respond to in the post-lesson interviews:

i) What experience have you had with teaching this topic?

ii) Have you used any different approaches/strategies in your teaching of this topic this time?

iii) What do you see are the most important considerations in planning to teach this topic to your students?

iv) What are some of the difficulties/misconceptions that some students have in relation to this topic?

v) How do you anticipate these?

Figure 3.1. Teacher interview schedule

The student focus-group interviews were conducted to generate data showing evidence of the teachers' PCK from the student perspective. Focus groups are a type of group interview that involve several participants, who, with the guidance of moderator (or researcher), discuss a topic (Denscombe, 2007). Focus groups provide an environment for rich interaction among participants and the researcher, because the ideas presented by the participant influence, and are influenced by, other members of the group (Denscombe, 2007). The focus-group interviews were semi-structured and involved a combination of pre-prepared questions and questions arising from the day's lesson observations. The pre-prepared questions (see interview schedule in Figure 3.2.) were designed to elicit responses from the students relating to their perception of their mathematics learning during the lesson and what actions their teacher had taken (e.g., demonstrations, examples, or explanations) that assisted them in their learning. Other interview questions arose from observations of specific teaching and learning interactions that occurred during the day's lesson.

Interview schedule for post-lesson interviews with consenting students

The student focus group interviews were up to 20 minutes in duration and included questions that arose from the following sources:

- His/her responses to the student survey for the day's lesson.
  The researcher's observations of particular teaching and learning interactions during the lesson.

Following are specific examples of the generic types of questions that students were invited to respond to.

i) How much did you know about this topic before class today?

ii) What do you know now?

iii) What happened in the lesson that particularly helped this knowledge growth?

iv) What are some of the effective things your teacher does to help you with your learning of these skills/concepts?

v) What was special about the examples chosen by the teacher that assisted you with your understanding of the problem?

Figure 3.2. Student focus-group interview schedule

3.7.3 Short Written Reflections

In addition to the focus-group interviews, short written reflections were also completed by participating students. These short reflections provided a brief but alternative source of evidence of the teachers' PCK from the students' perspective. The advantage of including this method of data generation was two-fold: the short reflections offered those who did not wish to participate in the interview the opportunity to contribute their perspective on their teacher's PCK, and, for students who did participate in the focus group, the short reflections offered the researcher additional prompts for discussion during the focus-group interviews.

The student short-reflection proforma (see Figure 3.3.) was used as a brief but additional format within which to generate data relating to PCK from the student perspective. The questions were designed to prompt students to reflect on and articulate aspects of their teacher's actions that particularly helped them with their mathematics learning during the lesson. Also, questions relating to the students' reflections could be used in the focus groups where a student may be invited to elaborate on his or her written response based on the day's lesson.

> **Student short-reflection**
> You are invited to provide written responses to the following two questions:
>
> What did you find to be the most helpful explanation, example or strategy that your teacher used in today's lesson?
>
> What did it help you to learn?

Figure 3.3. Student short-written reflection proforma

3.8 Procedures

A total of 18 lessons were observed and video-recorded by the researcher, including six lessons per class. Each lesson focused on one of two topic areas including calculus, or statistics and probability. Both topics were represented among the lessons observed for each of the three classes. The order in which the classes from School A (Mr Jones and Mr Taylor classes) covered these topics was as follows: differential calculus, probability, integral calculus. Mr McLaren's class (from School B), however, covered both differential and integral calculus before commencing the topic of probability.

An overview of the data generated from each of the three classes over a period of six lessons each is provided in Tables 3.2, 3.3, and 3.4. The lessons for Mr Jones and Mr McLaren's classes were scheduled either at the beginning of the school day up until recess, or during the period between recess and lunch. These lesson times were more convenient in terms of student availability for focus-group interviews immediately after the lesson.

In the case of Mr Taylor's class however, most lesson observations were scheduled in the afternoon after lunch. Therefore, the students had limited time to contribute focus-group or written responses due to buses and co-curricular schedules. As such, the students did not contribute written data for four of the six lessons. Instead, the students opted to meet with the researcher briefly at the end of each lesson prior to their departure.

3.8.1 Procedures Followed for Each Lesson

The researcher arrived to observe each lesson a few minutes before it commenced and positioned herself at a desk with the digital camera in view of the whiteboard. The students and teacher did not appear to take any notice of the video camera, particularly after the first visit, and seemed to carry out their teaching and learning activities as usual. The students were friendly and courteous towards the researcher.

The lessons were video-recorded by the researcher. Typically, each lesson began with an instructional phase during which the teacher would explain a concept or procedure using the whiteboard and/or demonstrate worked solutions to examples out of the prescribed text-book. Video footage of the lessons was obtained by the researcher using a hand-held digital camera while seated at a desk in the classroom, for the purposes of capturing multiple aspects of a teacher's PCK in action such as: classroom demonstrations, choice and use of examples, questioning, the use of representations, and responding to student ideas.

Given that the Mathematics Methods course was a high-stakes externally-assessed pre-tertiary course, it was important that the presence of the video camera was as unobtrusive as possible. Therefore, the researcher chose not to move around the classroom during the lessons to video-record other teaching and learning interactions such as the teacher's discussions with individuals or small groups. Occasionally, however, the researcher sought permission from individual students to obtain still photographs of their written work or their CAS calculator screen displays.

Five minutes prior to the end of each lesson all participating students were invited to complete the short reflection, and then at the end of each lesson some students participated in the post-lesson focus groups. The student post-lesson focus-group interviews took place immediately following each lesson and were up to 20 minutes in duration. The participants in each focus group self-selected based on their availability. There was greater variation in the composition of each focus group from Mr Jones' class than the other classes given that there were more students in his class. For Mr Taylor and Mr McLaren's classes, the same small group of students tended to volunteer to participate in the focus groups each time. Following the student focus-group interviews the teacher participated in a post-lesson interview, 20 minutes in duration.

Table 3.2

Mr Jones' class data

Date	**Class**	**Topic**	**Post-lesson teacher interview**	**Duration of video footage (minutes)**	**Number of students interviewed in focus group**	**Number of student questionnaires completed**
1st August, 2014	Differential Calculus	Applications of differential calculus including solving optimization problems	Yes	50	5	15/18
4th August, 2014	Probability	Introduction to the hyper-geometric distribution	Yes	45	4	7/18
8th August, 2014	Probability	Focus on variance	Yes	45	4	Time did not permit the completion of the short reflection proforma)
18th August, 2014	Probability	Revision of probability distributions	Yes	40	3	7/18
29th August, 2014	Integral Calculus	Calculating areas under curves	Yes	45	3	14/18
1st September, 2014	Integral Calculus	Calculating the area under graphs of functions	Yes	40	3	14/18

Table 3.3

Mr Taylor's class data

Date	**Class**	**Topic**	**Post-lesson teacher interview**	**Duration of video footage (minutes)**	**Number of students interviewed in focus group**	**Number of student questionnaires completed out**
5th August, 2014	Probability	Introduction to the hypergeometric distribution	Yes	40	4	3/10
11th August, 2014	Probability	Introduction to the normal distribution	Yes	40	3	4/10
12th August, 2014	Probability	The standard normal distribution	Yes	45	2	0
19th August, 2014	Integral Calculus	Introduction to anti-differentiation	Yes	50	5	0
25th August, 2014	Integral Calculus	Integration by recognition	Yes	35	3	0
28th August, 2014	Integral Calculus	Introduction to definite integrals	Yes	40	4	0

Table 3.4

Mr McLaren's class data

Date	Class	Topic	Post-lesson teacher interview	Duration of video footage (minutes)	Number of students interviewed in focus group	Number of student short-written reflections completed out
5th August, 2015	Differential Calculus	Rates of change	Yes	40	4	5/9
14th August, 2015	Integral Calculus	Anti-differentiation of exponential functions	Yes	45	6	5/9
28th August, 2015	Integral Calculus	Application to calculating areas enclosed by functions	Yes	45	4	4/9
16th September, 2015	Integral Calculus	Calculating areas between curves	Yes	40	5	5/9
17th September, 2015	Probability	Introduction to the binomial distribution	Yes	45	4	4/9
21st September, 2015	Probability	Applications of the binomial theorem	Yes	45	4	4/9

3.9 Data Analysis

Data analysis involved the ongoing process of transcribing, coding, and interpreting the multiple sources of data generated (i.e., video footage of lessons, transcribed teacher interviews and student focus-group interviews, and student short reflections), for each lesson.

3.9.1 Transcribing the Data

According to Bailey (2008), transcription is an important first step in data analysis because it involves the close examination of data, such as audio and/or video recordings, through "careful and repeated listening" and/or watching (p. 129). Bailey describes the process of transcribing data as an interpretive act rather than a technical procedure. Data analysis in this study involved the transcription of several data sources including: lesson observation and video footage, audio recordings of post-lesson interviews with teachers and students, and written data in the form of short reflections from the students.

Transcribing the Video Footage

Approximately 13 hours of video footage was produced across the three classes, not all of which was used in the analysis. It was not feasible to transcribe all the footage in its entirety. Instead, the researcher viewed the video footage of the instructional phases of each of the lessons and produced summaries of the footage. From these summaries, the researcher selected aspects of the video-footage to transcribe in full. This footage was chosen on the basis that it illustrated the range of PCK that was enacted by the teachers, and/or discussed from the multiple perspectives (i.e., the researcher, the teachers, and the students). The purpose of transcribing this footage in full, was to enable the construction of the Scenarios which are presented in Chapter 4. Each Scenario provides a detailed snapshot of the enactment of PCK from the perspectives of the researcher, the teacher, and students. Further detail about the nature and purpose of the scenarios are addressed in Sections 3.9.4 and 4.1.

Clarke (2013) acknowledges the role of data reduction within the context of classroom video study. He asserts that the researcher's "choice of classroom, the number of cameras used, who is kept in view continuously and who appears only given particular circumstances" all contribute to data reduction (p. 227). Clarke also emphases the idea that data reduction "does not stop" with the video recording but continues throughout the construction and coding of video transcripts given that the researcher has such a principal role in the construction of these data.

Each lesson summary was completed within days of collecting the video footage and presented in the format shown in Table 3.5. The preparation of the lesson summaries was time intensive and involved regular pausing and re-watching segments. In many cases the researcher needed to watch aspects of the footage multiple times as she interpreted the PCK in the teachers' actions.

Table 3.5

Excerpt from a lesson summary

Time	Transcript/outline Mr McLaren: Introduction to Differentiation by first principles.	Comments (preliminary coding)
0:30	T: "What do you understand the derivative to be?" S: Gives the rule for y=x^n (i.e. gives procedure rather than a meaning) T: [Recasts question]: What does the derivative tell us? S: The gradient at a point. (acknowledged by teacher as correct) T mentions that looked at limits last lesson	Questioning techniques Classroom techniques (of a lead-in focus question)
1:55	Emphasises that first principles differentiation is on the exam	Knowledge of Assessment
2:00	Refers to a diagram of an arbitrary function f(x), showing both a secant and a tangent through one particular point. Starts to label all the key points P (x, f(x)) and Q (x+h, f(x+h)) with discussion of the second point being h further along the x axis.	Knowledge of the standard diagram and explanation of first principles derivative

The preparation of the lesson summaries involved some initial descriptive coding as indicated in Table 3.5. According to Miles and Huberman (1994) early labels in the form of descriptive codes, which require little inference beyond the data itself, are particularly useful in "getting the analysis started, and in enabling the researcher to get a 'feel' for the data" (p. 176).

In relation to the interpretation of the video footage it is useful to draw upon Rowland's (2008) assertion that every "human account of events is an interpretation of the messenger/teller's experience" and that no "objective" account of a lesson can be written (p. 279). Therefore, to check for consistency of interpretation, video footage from at least one lesson per teacher was viewed independently by the researcher and at least one of her supervisors. The extent to which similar incidents (e.g., choice of particular example, acknowledgement) in the lesson were noticed by the researcher and her supervisor, were also of interest. There was general agreement in what was identified as PCK. Most discrepancies involved one viewer noticing one aspect that the other had not seen, rather than disagreeing on the category of PCK. In some cases, this was because it was possible to look at the teaching with different levels of "graininess". Given that the study was not attempting to quantify the occurrences of different PCK types, the researcher identified the most obvious examples rather than attempting to identify every micro-occurrence.

Transcribing Teacher Interview and Student Post-Lesson Interviews

As soon as possible after each lesson observation, the corresponding post-lesson teacher interview and student focus-group interview were transcribed verbatim for coding and analysis. The interviews and short written reflections were transcribed in full for the close examination and interpretation of evidence of PCK from the perspectives of the teachers and students.

The act of transcribing the data also afforded the opportunity to make initial connections between evidence of PCK observed by the researcher from the classroom observations and that discussed by the teachers and students.

3.9.2 Coding the data

Coding is an iterative process involving ongoing management and filtering to distil the "salient features" of the data for particular themes (Saldana, 2009, p. 9). The data generated in this study were coded manually using a combination of inductive (derived from the data) and deductive (derived from pre-existing theory and theoretical frameworks) processes. The inductive process involved the initial open coding of the data for the purpose, as described by Saldana (2009), of familiarising the researcher with the data and enabling deep reflection on its contents and nuances. It was appropriate to begin with an inductive approach to coding the data in order to remain open to possible emergent themes, even though the researcher was already sensitized to the elements of PCK evident in the existing body of literature.

The deductive phase of the coding process involved matching the codes obtained from the inductive coding process, with the components of the Knowledge Quartet and the Chick et al. PCK framework. These frameworks share some common components (e.g., knowledge of examples, anticipation of complexity, and knowledge of student errors) but they also offer uniquely different approaches to exploring PCK at the senior secondary mathematics level. The structure of the KQ lends itself to the examination of a teacher's PCK from different perspectives including: the PCK the teacher brings to his/her teaching (Foundation dimension); the PCK evident in the ways in the teacher make the content accessible to his/her students (Transformation and Connection dimensions); and a more dynamic kind of PCK evident in the teacher's unplanned moment-by-moment instructional decisions and actions in the classroom (Contingency dimension).

The Chick et al. framework offered a more fine-grained set of teacher knowledge components through which to explore teachers' PCK in action in the classroom. In addition, this framework offers scope to explore those aspects of PCK that derive from the application of general pedagogical knowledge within a specific mathematics content context. For example, the use of questioning – a generic classroom technique – is transformed into PCK when used to focus on specific mathematics content for students (e.g., "Because we're looking for a minimum, what will we eventually need to find, somewhere during this question?").

The Chick et al. framework also includes categories associated with the depth and connectedness of mathematics (i.e., *mathematical structure and connections*, and *deconstructing mathematics into key components*) which are particularly applicable in the senior secondary mathematics teaching context. While the KQ also includes categories related to making mathematical connections (i.e., *connections between procedures* and *connections between concepts*), the kind of mathematical connections demonstrated by the senior secondary mathematics teachers in this study involved helping students to recognise mathematical properties in specific situations based on broader generalisations. Therefore, it was more useful to explore these connections through the filter of *mathematical structure and connections* and *deconstructing mathematics into key components* from the Chick et al. framework rather than attempting to distinguish, as the KQ does, between connections involving procedures and those involving concepts.

Table 3.6 lists, describes, and exemplifies each code used to analyse the multiple sources of data in this study, and is structured to show how clusters of inductive codes were aligned to their corresponding deductive code derived from the KQ and/or the Chick et al. framework.

Table 3.6

Inventory of codes used in the analysis of data

Open Codes (Inductive coding process)	**Codes derived from KQ and Chick et al. framework (Deductive coding process)**	**Description**	**Example**
Beliefs about students' learning	Theoretical underpinning of the pedagogy (KQ)	Teacher expresses belief about how students best learn mathematics	"The girls on the other hand who sat on the other side of the room who did a lot of questioning, talking and bringing things out all got through [the exam]" (Teacher interview comment)
Beliefs about teaching		Teacher expresses belief about effective ways to teach mathematics	"So, let them make the mistake … it's not best just to give it to them but try to lead them in the right direction." (Teacher interview comment)
Goals of mathematics teaching/learning		Teacher expresses belief about what the key purpose of learning/teaching mathematics	"the focus of the understanding comes from like what we did today with identifying which distribution to use … you are focussing on getting to the end product. Maybe that's what is important in the sense that out in the real world it's about what tool I am going to use. (Teacher interview comment)

Open Codes (Inductive coding process)	Codes derived from KQ and Chick et al. framework (Deductive coding process)	Description	Example
Common errors	Knowledge of student errors (both frameworks)	Teacher discusses/identifies common mathematical errors made by the students	“The biggest difficulty with people who are struggling a bit and don’t quite know how to integrate, so they fall back on their differentiation techniques … so they’ll add one to the index but then multiply by the new index [instead of divide]” (Teacher interview comment).
Focuses on procedures	Concentration on procedures (KQ)	Teacher focuses on unpacking mathematical processes in detail	Teacher provides detailed step-by-step instructions of the keystrokes needed to use integral calculus to find the area enclosed by $y = \sin 2x$ over a specific domain using the CAS calculator (Lesson observation)
Textbooks	Adherence to textbook (KQ)	Teacher focuses on the way in which a textbook addresses a particular mathematical idea	Teacher urges students to adhere to the convention used in the prescribed textbook – if the question asks for “an” antiderivative then the answer should be expressed with an arbitrary constant of integration of zero (Lesson observation)

Open Codes (Inductive coding process)	**Codes derived from KQ and Chick et al. framework (Deductive coding process)**	**Description**	**Example**
Use of content knowledge	Overt display of subject matter knowledge (KQ)	Teacher draws upon content knowledge in ways that go beyond that required to solve a particular problem	Teacher decides to sketch a general log graph to enable the students to see what a particular log value was negative, even though the problem did not require the graph as part of its solution (Lesson observation)
Emphasises notation	Use of mathematical terminology (KQ)	Teacher places special emphasis on the students' use of mathematical notation	Teacher focuses on the importance of the correct use of the operators $\int$ and dx when anti-differentiating expressions (Lesson observation)
Exam	Knowledge of assessment (Chick et al. framework)	Teacher discusses requirements of the final examination	"I don't know how many times in exam situations where you're asked to find k somewhere across lots of topics" (In class comment by teacher)
Explanation/worked solution	Teacher Demonstration (KQ)	Teacher demonstrates a worked solution to a problem or explains a mathematical process or idea.	The explanations and worked examples on the board with consistent pausing to further explain items helped me gain an understanding of the work. (Student written response)

Open Codes (Inductive coding process)	Codes derived from KQ and Chick et al. framework (Deductive coding process)	Description	Example
Choice/use of examples	Knowledge of examples (both frameworks)	Teacher discusses choice/use of standard text book examples.	"I wanted to choose some examples that concentrated a bit more on trig [trigonometry] and brought in some other skills from earlier in the year because the thing with the Methods course is that Trig pops up everywhere in lots of topics" (Teacher interview comment).
		Teacher discusses choice/use of instructional example	"I wanted to have enough values for x but not too many, like *x* from 1 to 30. I didn't want to have that many values but then I didn't want to have too few values because then the spread wouldn't be as obvious" (Teacher interview comment).
Uses/discusses teaching materials	Knowledge of instructional resources (both frameworks)	Teacher uses or discusses teaching resources (particularly the prescribed textbook and the CAS calculator)	The most useful thing [in the lesson] was the demonstration on how to use the calculator to find areas (Student written response) "The focus today was very much on the calculator. Because I know that to work out more complex areas, I mean, it is exam driven a bit, but for the more complex ones they are going to be asked to sketch it on the calculator" (Teacher interview response).

Open Codes (Inductive coding process)	**Codes derived from KQ and Chick et al. framework (Deductive coding process)**	**Description**	**Example**
Uses/draws graph/diagram	Knowledge of representations (both frameworks)	The teacher uses a graph or diagram to illuminate a particular idea	Teacher constructed the graphs of two strategically chosen probability distributions to illuminate the idea of the spread of the data (lesson observation).
Cognitive demand	Anticipation of complexity (both frameworks)	The teacher discusses or attends to the cognitive demand of a specific task	"He [Mr McLaren] must understand what's stumping us, what the small little part of the question that's stumping us" (Student, focus-group interview)
Connections	Mathematical structure and connections (Chick et al. framework)	Teacher draws connections between mathematical ideas by helping students to recognise mathematical properties in specific situations based on broader generalisations.	"Often people tune in and look for 9 and 25 and 36 as so on. But that difference of perfect squares [method of factorisation] can be used for any number." (In-class comment by teacher)
Absence of connections	*Absence* mathematical structure and connections (Chick et al. framework)	Teacher overlooks an opportunity to address specific connections.	"I don't think spending quite a large amount of time on it [the Fundamental Theorem of Calculus] is needed. And I doubt even if I did that whether there would be full understanding about how the theorem works, or whether it's worth the time and the effort" (Teacher interview)

Open Codes (Inductive coding process)	Codes derived from KQ and Chick et al. framework (Deductive coding process)	Description	Example
Unpacks ideas	Deconstructing mathematics into key components (Chick et al. framework)	Teacher unravels or attempts to unpack a mathematical idea or solution process	Teacher attempts to unpack why the the ‘b’ “disappears” in the formula Var($aX + b$) = a^2Var (X) (Lesson observation)
Respond to students’ thinking	Responding to students’ ideas (both frameworks)	Teacher’s response to student’s unexpected question or idea	Teacher acknowledges but puts aside a student’s suggestion of an alternative method of solution to a standard optimisation problem (Lesson observation)
In-the-moment decision	Teacher insight (KQ)	Teacher reflects-in-action and makes in-the-moment decision to develop students’ understanding of a mathematical idea	Teacher made an in-the-moment decision to sketch the graph of $y = \log_a x$ to enable students to see that $\log_e\left(\frac{8145059}{8145060}\right)$ has a negative value (Lesson observation)
Teaching tool	Classroom techniques (Chick et al. framework)	Teacher uses generic teaching techniques such as questioning or encouraging students to think through a mathematical idea for themselves.	Yeah getting you to nut it out a bit before he prompts you which is probably a good thing … he’ll like say a couple of things like “what do you do with this?” and then everyone’s like “Oh yeah” And from there, like that one little prompt, sometimes we can just go through and do the entire question by ourselves (Student focus-group response)

The following subsections focuses on issues of inter-coder reliability followed by descriptions of the ways in which each data source was coded.

Inter-coder Reliability

To ensure inter-coder reliability, the researcher and her two supervisors independently coded a lesson summary, a teacher interview transcript, a student focus-group interview transcript, and a set of short-reflections from each of the three participating classes. It was important to determine the extent to which the three researchers noticed similar aspects of PCK evident in each lesson summary and assigned codes accordingly. In addition, the researcher and her supervisors collaborated in the process of aligning the inductive codes to existing categories in the two theoretical frameworks (i.e., the KQ and the Chick et al. framework).

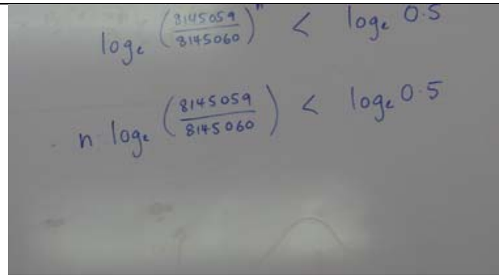
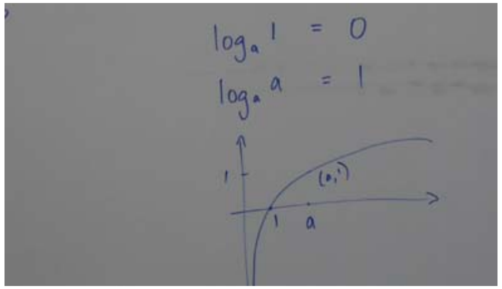
As part of the process of checking for inter-coder reliability, the researcher and her supervisors discussed complexities in relation to assigning codes. For example, the code *concentration on procedures* was assigned to instances where a teacher focused on the steps involved in solving a mathematical procedure. If, however, this focus on procedures was also associated with a lack of attendance to a key mathematical concept then the code *mathematical structure and connections* (from the point of view of an absence of connections) was also assigned. In other words, the idea of coding an aspect of data from the perspective of the absence of an element of PCK, was one of the complexities of coding the data identified by the researcher and her supervisors

Coding the Lesson Summaries

Each lesson summary was coded inductively as shown in the example featured in Table 3.7. The inductive codes were then compared against the categories of the KQ and the Chick et al. framework. The frequency with which the inductive codes were assigned in each lesson summary was dependent upon the teaching and learning interactions taking place at any given time. In many cases, based on the PCK enacted by the teacher, several codes were assigned over small sections of data. For example, Table 3.7 indicates that multiple codes were assigned to data arising from just three minutes of video footage.

Table 3.7

Excerpt from coded lesson summary

Duration	Description	Accompanying whiteboard photos	Initial open codes (inductive coding)	Codes from frameworks (deductive coding)
11:05-11:10 am	Teacher encourages students to recognise "what to do next". Someone says divide by log 0.5. Teacher reminds them they need *n* on its own so need to divide by log of the other fraction. Teacher asks what happens when you divide by the log of the fraction. Someone mentions inequality sign change, but no one knows why and some say it doesn't change. Teacher says it does change and use questioning to get them to recognise why.		Use of funnellingquestions	Classroom technique (Chick et al. framework)
	Students not forthcoming with reason so teacher puts $\log_a 1$ on the board but then rubs it off and ponders for a few seconds and then tells them "remember the log graph" and it will be easiest way to see (why inequality sign changes). Draws attention to the key points ($x = 1$ etc). Makes the point that it can be any base. A student exclaims "Oh so that's below 1 so it's a negative so that's why you change it around"		Responds on the spot to students' lack of understandinng	Teacher insight (KQ)
	Teacher acknowledges student's correct response and highlights any value of log *x* between zero and one is negative. There are a several audible comments like "Oh yeah" and "I get that".		Sketches graph to illustrate an idea	Knowledge of representations (both frameworks)
	Mr McLaren points to the 8145059 over 8145060 and reiterates that it is less than 1 so its log will be less than 1 – hence the sign change when divide by log of that negative number.		Shows connections between ideas	Mathematical structure and connections (Chick et al. framework).

Coding the Teacher Post-lesson Interview Transcripts

Each interview transcript was coded manually within days of transcription using a combination of inductive and deductive coding. Excerpts from two different coded interview transcripts are shown in Table 3.8 and Table 3.9.

Table 3.8

Excerpt of coded interview transcript (Mr Taylor)

Question/response	Inductive code	Code from framework
Mr Taylor: I first taught this topic in 1977, I think. I think the advantage of teaching it a large number of times is that you get to know where kids are likely to run into trouble and when they are going to make an error, and you can see in advance what kinds of errors they are going to make. **Researcher:** What kinds of errors in particular?	Common errors	Knowledge of student errors (both frameworks)
Mr Taylor: The main trouble is not sorting out the differences between differentiation and integration - that's the biggest difficulty with people who are struggling a little bit and don't quite know how to integrate so they try and fall back on their differentiation techniques.	Common errors	Knowledge of student errors

Table 3.9

Excerpt of coded interview transcript (Mr Jones)

Question/response	Codes (inductive)	Code(s)
Mr Jones: I wanted to give them an example of one that didn't require use of the calculator at all, because the nature of our course is that there is a calculator and a non-calculator section of the exam. So that was an important example because, I mean, I don't want to get too caught up in the exam, but in reality, I have to be faithful to anticipating what sort of questions come up.	Exam	Knowledge of assessment (Chick et al. framework)

Coding the Focus-group and Short Reflection Transcripts

The student post-lesson focus-group interview responses and short reflections were interpreted and coded based on evidence that could be inferred about their teacher's PCK. Examples of excerpts from two coded focus-group transcripts are shown in Tables 3.10 and 3.11, and several coded short reflections corresponding to the transcripts are shown in Table 3.12. Note that the coding for the short reflections referred to the first question because the second question (What did it help you to learn?) was designed to elicit more detail from the students in relation to their response to the first.

Table 3.10

Excerpt of coded focus-group interview transcript (Mr Jones' students)

Question/response	**Inductive code**	**Code from frameworks**
Simon: I think it probably helped as well that he [Mr Jones] was actually doing the questions on the board instead of just asking us which one it was and then going on to the next one, so he actually showed us how to work each one out. I think that's what made it click for me after the second or third one [question] I started to know which one [probability distribution] was which.	Worked solution	Teacher Demonstration (KQ)
Danny: When the flow chart was up there, and it had the two headings binomial and hyper geometric then it said underneath it what you had to look for to try and distinguish which was which. And then we did the questions and you could see what you had to look for in the questions.	Generic technique	Classroom Technique (Chick et al. framework)

Table 3.11

Excerpt of coded focus-group interview transcript (Mr McLaren's students)

Question/response	Inductive code	Code from frameworks
Researcher: You also recognized the link to Pascal's Triangle Kale, at what point did you realise that?		
Kale: When the second 4 went down I was like "Oh" and then I was thinking there was a kind of pattern and I was sort of thinking because we touched on it on Pascal's Triangle for something else a while ago and it kind of became obvious. I think the way that he [Mr McLaren] kind of sort of left it and didn't say it out loud or tell us. He almost let us discover it for ourselves which kind of imprints it in your mind in a strong kind of way.	Connections Generic technique	Mathematical structure and connections (Chick et al. framework)Classroom Techniques (Chick et al. framework)
Grace: Like we'd done the binomial theorem originally like earlier in the year, and now it's sort of got another use for it so you can relate our other work to this, it's not just a random section of work anymore.	Connections	Mathematical Structure and Connections (Chick et al. framework)

Table 3.12

Excerpt of coded short-written reflections (Mr McLaren's students)

Response Q 1 What did you find to be the most helpful explanation, example or strategy that your teacher used in today's lesson?	**Response Q 2** What did it help you learn?	**Inductive code**	**Code from frameworks**
I appreciated the step by step instructions of how to do it and how there were plenty of examples both easy and difficult.	How to actually set the questions out and how to do the simple questions and the more complicated ones.	Explanation	Teacher demonstration (KQ)
How he encouraged us to solve the questions ourselves and nudged us along when we were stuck, instead of straight up giving us the answers.	The methods of doing the various types and difficulties of questions.	Teaching tool	Classroom Technique (Chick et al. framework)

3.9.3 Analysing PCK from the Perspective of Multiple Data Sources

Mason (2002) emphasises that social phenomena are multi-dimensional, so it follows that studies of such phenomena should seek to "grasp more than one of those dimensions" (p. 191). This approach relates to the idea of triangulation in its broadest sense, where a combination of methods are used to explore a set of research questions from different angles and in a multi-faceted way (2002). The value of triangulation lies in providing evidence that enables the researcher to construct quality explanations of the social phenomena under study, irrespective of whether the different data sources corroborate or contradict each other (Mathison, 1988). To this end, PCK in the context of the senior secondary mathematics classroom was explored from multiple perspectives, including that of the researcher, the teachers, and their students, using several data sources (lesson observation, interviews, and written responses).

As discussed in previous sections, data generated for each lesson included a lesson summary (prepared from the video footage), a post-lesson teacher interview transcript, a student focus-group interview transcript, and the students' short reflections. Following the coding of the data, the three sources of evidence of PCK generated for each lesson were examined for commonalities and differences. These comparisons were made by exploring the extent to which the researcher's perspective (embodied in the lesson summary arising from the observations and video recording), the teacher's perspective (provided in his post-lesson interview) and the students' perspectives (conveyed in the focus-group interview and short reflections) corroborated each other. For instance, the researcher's observation of a teacher's salient use of a specific example during a lesson might align with both the teacher's justification for selecting the example, as well as the students' perception of the usefulness of that example.

Other cases, however, may have shown little alignment between the researcher's observation of a teacher's PCK enacted during a specific lesson event, and the PCK that the teacher discusses and attributes to his instructional decision. Similarly, a teaching and learning event that is considered by the researcher to be particularly noteworthy in relation to the teacher's PCK may not be noticed or discussed by the students in their focus-group or short reflections. Exploring this variation into the extent to which the three data sources corroborated each other enabled a broader examination of PCK at the senior secondary mathematics. As highlighted by Mason (2002) a rich and holistic understanding of any multi-dimensional phenomenon may be gained by exploring the phenenomenon from multiple and sometimes conflicting perspectives.

3.9.4 Data Reduction and Display

Miles and Huberman (1994) emphasise the role of data reduction and display in the analysis of qualitative data. Data reduction is an integral part of data analysis and involves the process of reducing a high volume of data without losing crucial information or stripping the data from its context (1994). As described by Miles and Huberman, data reduction occurs throughout all phases of the data analysis process including the initial stages of summarising the data (e.g., producing summaries of transcripts), the coding process, and in later stages of analysis during which the data are explained and conceptualised.

Closely associated with data reduction are data displays, which are used at all stages of data generation and analysis, to compress and assemble information in organised ways (Miles & Huberman,1994). Data displays represent the different iterations of the data analysis process and provide the basis for further analysis (1994).

In this study, the data were initially displayed in the form of the lesson summaries, interview transcripts, and transcripts of the students' short-reflections. Following the coding of the data, the next phase of data reduction and display involved aligning the teaching and learning event described in each lesson summary, with corresponding teacher and or student responses. For example, Table 3.13 provides a snapshot of data from a lesson involving the application of differential calculus. It shows the alignment between the teaching and learning event, Mr Jones' interview response, and a student's short reflection response. Note, however, that in other instances there was variation in the extent to which the three data sources corroborated each other.

Table 3.13

Alignment of teaching event with teacher and student data

Teaching event	**Codes**	**Teacher data (post-lesson interview)**	**Codes**	**Student data (interviews and reflections)**	**Codes**
Mr Jones encouraged students to recognise the key processes involved in solving optimisation problems by directing their thinking through questioning: "Asked funnellingquestions such as Because we are looking for a minimum, what are we going to have to find eventually somewhere in this question?	Teacher Demonstration Deconstructing mathematics into key components Classroom Techniques	**Mr Jones:** This year I've probably identified key words in the question and making sure that they understand what the process is. When you are teaching a topic like that, these are the key steps you've got to do. So, give them the framework I suppose and hopefully they can apply that framework to understanding other situations. This year I've really concentrated on that.	Deconstructing mathematics into key components	"The examples on the board helped me recognise when to do what (e.g., "when to look for a minimum or maximum and making d′(x) = 0"). (Lucy; survey).	Teacher Demonstration (KQ) Deconstructing mathematics into key components (Chick et al. framework)

The next table (Table 3.14) shows how the second key iteration of data reduction and display involved summarising the previous displays into a matrix that briefly described key teaching and learning events for each lesson and any corresponding responses from the teacher and students.

Table 3.14

Excerpt from matrix showing summary of key teaching and learning events across all lessons

Teacher	**Topic**	**Focus of the lesson**	**Key teaching and learning events in the lesson**
McLaren	Probability	Introduction of binomial distribution and link to binomial theorem Pascal's triangle.	McLaren gradually unravelled pattern that followed Pascal's triangle and then teacher linked to binomial theorem. Students noticed Pascal's triangle it as it unfolded. Both teacher and student talked about "discovering it for themselves." Mr McLaren revealed during interview that he had used a table from the prescribed textbook as the basis of this activity.
McLaren	Probability	Focus on the binomial distribution and solving textbook exercises	Teacher draws the graph of $y = \log_a x$ to show why a specific log value is negative and hence inequality sign must be changed upon division. Students commented that this helped them to see why – and strengthened log understanding.

The researcher then selected, from this matrix, specific teaching and learning events for the construction of the Scenarios presented in Chapter 4. These teaching and learning events were selected because they were illustrative of the range of different elements of PCK enacted by the teachers in this study, and/or they exemplified instances of PCK from the perspective of the researcher, the teacher, and or the students. The lesson excerpts, along with any corresponding teacher and student data were transcribed in full, to achieve the level of thick description (e.g., Geertz, 1973) provided in the Scenarios presented in Chapter 4. The purpose of the Scenarios was to capture and present, in detail, the teaching and learning interactions between the teachers and their students. Further details of the nature and purpose of the Scenarios is discussed in Section 4.1.

3.10 Addressing Issues of Trustworthiness

Ascertaining the trustworthiness of a qualitative research investigation encompasses issues of reliability and validity, but in fundamentally different ways to quantitative studies. From a quantitative research perspective, reliability is commonly described as the extent to which the results of a study are replicable or repeatable (e.g., Joppe, 2000). Internal validity is about ensuring that a study measures what is actually intended, and external validity is concerned with whether or not the findings of one study can be applied to other studies (Merriam, 1995).

The qualitative research paradigm is underpinned by the idea that there are multiple interpretations of reality. As such there are inherent complexities associated with addressing issues of trustworthiness in qualitative research studies (e.g., Agar, 1986; Guba, 1981; Guba & Lincoln, 1981; Merriam, 1995). It follows therefore that ideas of validity and reliability take on different meanings within the context of qualitative research. Researchers including Agar (1986), and Guba and Lincoln (1981) make the case that the different conceptualisations of validity and reliability in qualitative research warrant different terminology. For example, Guba and Lincoln (1981) use the terms *credibility* (in place of internal validity), *transferability* (instead of external validity), and *dependability* (instead of reliability). Within these three criteria, Guba and Lincoln propose a range of strategies that may be used to enhance the trustworthiness of a qualitative research study. Following on from the work of Guba (1981) and Guba and Lincoln (1981), others including Merriam (1995) and Shenton (2004) have developed a repertoire of strategies to address issues of credibility, transferability, and dependability. Some of these strategies were used to ensure the trustworthiness of this study and are discussed in the following subsections. Note that in some cases there is overlap among strategies. For example, triangulation is applicable to both credibility and dependability although the focus is slightly different for each.

3.10.1 Credibility

According to Merriam (1995) internal validity (credibility) is concerned with the extent to which one's research findings are "congruent" with reality (p. 53). Given that qualitative research is underpinned by the idea that there are multiple interpretations of reality, there are unique complexities associated with addressing the credibility of studies of this kind. Guba (1981) suggests that the credibility of qualitative research is concerned with acknowledging and accounting for complexity, rather than fixing variables and guarding against "sources of error", as is commonly the case within the positivist research paradigm (p. 84). The following strategies were used in this study to address issues of credibility.

Prolonged Engagement at a Site

Guba (1981) asserts that prolonged engagement at the site is an important strategy for addressing the credibility of a study because it minimises possible "distortions" that may stem from the researcher's presence (p. 84). The strategy is also concerned with ensuring adequate time for the researcher to grow in his/her role as the key instrument of data generation, and for the participants to adjust to the researcher's presence (1981). In this study, the researcher sought to adopt a balance between spending extended time in the participating classrooms and respecting the high-stakes nature of Mathematics Methods as a pre-tertiary course. Achieving this balance was important because stakeholders (e.g., school, teachers, students, and parents) would expect minimal interruption to the regular teaching and learning environment. The researcher observed six lessons for each class, usually one to two lessons per week over the duration of a school term, with no visits at least two weeks before the end of term when students were engaged in examination preparation. This level of engagement on site afforded the researcher the time to generate rich data showing evidence of the teachers' PCK in a range of contexts.

Peer De-briefing

Guba (1981) also identifies peer debriefing as a valuable opportunity for researchers to "test their growing ideas and to expose themselves to searching questions" (p. 85). Similarly, Merriam (1995) recommends that researchers ask colleagues and peers to examine the data and provide feedback on the credibility of emerging findings. Throughout the data generation and analysis stages of this investigation, the researcher and her two supervisors engaged in regular meetings to examine and discuss the data and emerging findings in relation to PCK. These meetings also included checking for inter-coder reliability in relation to each of the data sources (discussed in Section 4.10.3).

Tactics to Ensure Honesty in Informants

"Tactics to help ensure honesty in informants" is one of the provisions, identified by Shenton (2004), that enable researchers to "promote confidence that they have accurately recorded the phenomena under scrutiny" (p. 64). Shenton highlights the importance of establishing a rapport with informants (participants) from the outset to ensure they feel comfortable to offer frank and open responses, particularly in an interview situation. The idea that participants are more likely to offer honest responses when there is a relationship of trust between the researcher and his/her participants is highly relevant to this study.

As a teacher of secondary mathematics herself, the researcher appreciated the sensitivities associated with discussing and having one's practice observed and video-recorded. According to Lasagabaster and Seirra (2011) it is common for many teachers, even those who are experienced, to feel a sense of uneasiness about having their practice observed. Lasagabaster and Seirra identify trust and mutual respect between the researcher and the teacher as key attributes of successful lesson observation. The researcher in this study considered her role as researcher in this study as an opportunity for her own professional learning, given that she had no previous experience with teaching the MTM315 course. As such, the relationship between researcher and teacher was positive and collegial, which in turn enabled rich and open dialogue during the post-lesson interviews.

Background, Qualifications, and Experience of Researcher

Shenton (2004) asserts that the background, qualifications, and experience of the researcher impact on the credibility of a qualitative study. The researcher in this study is an experienced secondary mathematics teacher who majored in mathematics as part of her undergraduate degree in Education. She therefore possessed the knowledge and experience to notice and interpret evidence of PCK both in action in the classroom, and in responses provided by teachers and students.

3.10.2 Dependability

Dependability in qualitative research relates to whether, or not, the results of a study are consistent with the data generated (Merriam, 1995). Guba (1981) draws attention to the complexities relating to issues of dependability in qualitative studies including the presence of apparent inconsistencies stemming from "developing insights" on the part of the researcher (p. 86). Drawing on the work of Guba and Merriam, the researcher used the strategies of peer de-briefing and an audit trail to address issues of dependability in this study.

Peer De-Briefing

While peer de-briefing was used to enhance the credibility of this study, it was also a useful strategy to address dependability. This strategy enabled the researcher to check that her interpretation of the emerging findings was reasonable and consistent with the data generated.

Audit Trail

An audit trail, as discussed by Guba (1981) and Merriam (1995) comprises a researcher's detailed description of how data were generated and interpreted including the emergence of codes and themes. Details of the ways in which data showing evidence of PCK were generated, coded, and interpreted are discussed earlier in the chapter.

3.10.3 Transferability

Guba (1981) alludes to the idea that the kind of generalizability associated with quantitative studies is not applicable in the qualitative paradigm, given that most social phenomena are context bound. It is therefore impossible to make statements that are applicable to all situations. Instead, the qualitative researcher seeks to provide descriptive and interpretive statements of a particular context. Drawing on the work of Guba (1981) and Merriam (1995), the strategies used in this study to address issues of transferability include thick description and multi-site design.

Thick Description

It is widely documented that *thick description*, based on the seminal work of Geertz (e.g., 1973), is one of the ways in which a researcher promotes the credibility of a qualitative study (Creswell & Miller, 2000; Denzin, 1989; Huberman & Miles, 2002). Thick descriptions are deep and detailed accounts of the phenomenon or situation under study (Denzin, 1989). Creswell and Miller (2000) describe how the purpose of a thick description is to create a written account that enables the reader to feel that "they have experienced, or could experience, the events being described in a study" (p. 129).

Thick description was achieved in this study by transcribing selected lesson events in full, from the video footage, to capture in detail the teaching and learning interactions between the teachers and their students. In addition, all interviews and written data were transcribed verbatim to reflect the voices of the participants as closely as possible. The presentation of the results, in the form of scenarios featured in Chapter 4, are also characterised by thick descriptions. Each scenario depicts the voices of the participants within the context of descriptions of specific teaching and learning events.

Multi-site Designs

Merriam (1995) draws upon the seminal work of Glaser and Strauss (1967) who highlight that transferability is enhanced by the use of several research sites or cases, especially those representing variation. Among the three cases studied in this research project, two different school sectors were represented (Catholic and Independent). The researcher attempted to include representation by the Government Sector as well, but unfortunately the consenting teacher had to withdraw before the study commenced for health reasons.

3.11 Summary of the Chapter

This chapter has explained the design and implementation of this collective case study into senior secondary mathematics teachers' PCK. In summary the chapter has presented:

- The qualitative research perspective underpinning the study design
- The collective case selection
- Ethical considerations including the processes involved in recruiting participants
- Details of the teacher and student participants
- Justification and explanation of the data generation methods
- Procedures
- Data analysis including the ways in which PCK were analysed from multiple perspectives
- Issues of trustworthiness of the study

The next chapter presents the results of the data analysis in the form of 14 scenarios each consisting of a teaching and learning event from a specific lesson, corresponding responses from the teacher and/or students, and a commentary highlighting evidence of aspects of the teachers' PCK.

Chapter 4

Results

4.1 Introduction

The results presented in this chapter are qualitatively described in a series of 14 scenarios. Each scenario consists of a teaching and learning episode from a specific lesson, corresponding teacher and student data, and a commentary highlighting key elements of the teachers' PCK evident in the lesson. That is, each scenario presents insights from three perspectives, that of the researcher (in the form of the lesson excerpt), the teacher, and his students. The PCK is identified against elements of the Knowledge Quartet (KQ) and the Chick et al. framework (PCK framework).

Data featured in the scenarios were selected on the basis of one, or a combination, of the following criteria: the data were typical of the ways in which PCK was enacted and or discussed by the teachers and students, the data showed evidence of PCK that did not feature prominently in the study but was nevertheless noteworthy, the data illustrated interesting tensions relating to PCK in the senior secondary mathematics classroom (e.g., the perceived dichotomy between addressing mathematical ideas in depth and meeting the requirements of a content dense curriculum). Collectively the scenarios include the enactment of various aspects of PCK by three teachers. It is important to highlight that Mr Taylor's class featured in only one of the 14 scenarios because limited student data were generated due to the circumstances explained in Section 3.8. Mr Taylor's contribution to the analysis was notable for his selection and sequencing of examples to illuminate skills and concepts. Each scenario is introduced by a table providing the rationale for the data selected for the construction of each scenario and an overview of the key aspects of PCK identified.

4.2 Scenario 1: "Tom's Suggestion"

Table 4.1

Information relating to Scenario 1

Date of Lesson	1st August 2014
Teacher	Mr Jones
Topic	Applications of differential calculus to solving optimisation problems
Rationale for selecting the data used to construct this scenario	Scenario highlights Mr Jones' response to a contingent teaching opportunity triggered by a student's suggestion of an alternative method of solution. Scenario depicts the typical way in which this teacher demonstrated worked solutions to routine problems
Elements of PCK	<u>Concentration on procedures</u> Foundation (KQ) <u>Knowledge of assessment</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Teacher demonstration:</u> Transformation (KQ) <u>Knowledge of examples:</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Classroom techniques:</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Responding to students' ideas:</u> Contingency (KQ) <u>Mathematical structure and connections (absence of)</u> Content knowledge in a pedagogy context (Chick et al. framework) <u>Deconstructing mathematics into key components:</u> Content knowledge in a pedagogy context (Chick et al. framework)

This scenario is based on a lesson focused on solving optimisation problems. Optimisation problems are the key focus of applications of differential calculus in the Mathematics Methods course and involve practical situations in which students are required to minimise or maximise a quantity. The teaching episode features the teacher's worked solution to the "distance problem" shown in Figure 4.1.

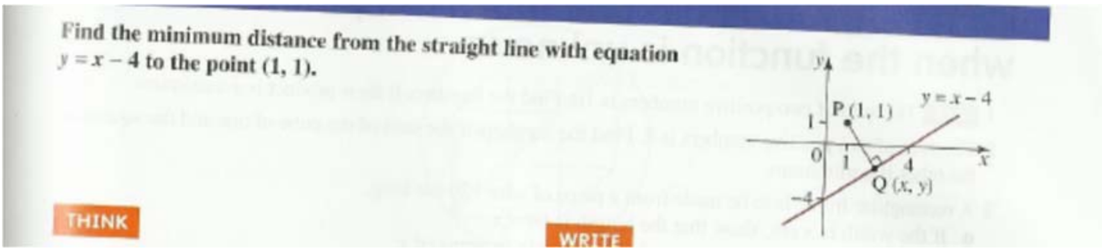


Figure 4.1. An example of a minimum problem when the function to be minimised is not given directly (Hodgson, 2013, p. 381)

4.2.1 Lesson Excerpt

Mr Jones began his demonstration by reproducing the diagram shown in Figure 4.1. There was no mention that the minimum distance is necessarily the perpendicular distance between point P and the line $y = x - 4$. The method of solution involved developing the appropriate distance function, finding its derivative, and then equating the derivative to zero to locate the x coordinate that would give the minimum distance. Mr Jones spent a few minutes reacquainting the students with the formula for the distance between two points, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. He then emphasised the idea that the point Q (x, y) (see Figure 4.1) must be expressed in terms of x only, that is $(x, x - 4)$, in order to develop a distance function with respect to x. The distance function $d(x) = \sqrt{2x^2 - 12x + 26}$ was obtained using the coordinates of points P (1,1) and Q (x, y) (see Figure 4.1).

Mr Jones encouraged the students to recognise the key processes involved in solving the optimisation problem by highlighting key words in the question (e.g., minimum), and by using funnelling questions (e.g., Bauersfeld, 1994).

Mr Jones: Because we are looking for a minimum, what are we going to have to find eventually somewhere in this question? [Class response: the derivative] Yes, and then we make the derivative equal to? [Class response: zero] Good and then solve for? [Class response: *x*] Good. And that should be an automatic reaction when we see the word "minimum" or "maximum"; [it] should be our trigger to say "right, that's our process".

The teacher's demonstration of the solution to the "distance problem" is continued in the following excerpt from the lesson:

Mr Jones: Before I can get the derivative of this function [points to the function $d(x) = \sqrt{2x^2 - 12x + 26}$]] what form do I need to put it in Jessie? It's in surd form at the moment.

Jessie: In power form.

Mr Jones: That's right, power form [rewrites the function as $d(x) = (2x^2 - 12x + 26)^{\frac{1}{2}}$].

Ok so to find the derivative $d'(x)$, what comes out the front Angela?

Angela: Um a half.

Mr Jones: That's right $\frac{1}{2}$, and then we multiply by what, Dyan?

Dylan: Oh, the derivative of the bracket.

Mr Jones: Yes. The derivative of the bracket which is $(4x - 12)$ and then multiplied by? What's the last bit Harry?

Harry: Um the brackets to the power of negative a half.

Mr Jones: Yes good [completes the differentiation process to yield:

$d'(x) = \frac{1}{2}$ (4x-12) $(2x^2 - 12x + 26)^{-\frac{1}{2}}$]. Are we all right with that? There's your process. Ok so we've got $4x - 12$ in the numerator and in the denominator, we've got the 2. Remember that your negative a half [points to the expression $(2x^2 - 12x + 26)^{-\frac{1}{2}}$] moves to the denominator so we have $d'(x) = \frac{(4x-12)}{2\sqrt{2x^2-12x+26}}$. Now tell me if I've done too many steps at once there? Ok so the $(4x - 12)$ and the 2 have stayed where they were and the bracket to the negative a half has gone underneath. Then I've just changed it from the power of a half to the square root.

The remainder of the solution process involved equating the derivative to zero to yield the value of x which gives the minimum (i.e., $x = 3$), and then substituting this value into the distance function to find the minimum distance.

Mr Jones: Now remember you haven't finished once you've found that $x = 3$ and this can be a trap for losing marks. We need to substitute back into the original function. [Following some further discussion with the class, Mr Jones records the final part of the solution on the board and concludes that the minimum distance $d\,(3) = \sqrt{8} = 2\sqrt{2}$]

He then highlighted the problem as a "classic example" of the kind of question which may appear in the non-calculator section of the examination.

Mr Jones: Have we needed our calculator at any stage so far? No and we don't need the calculator for this, not even to change $\sqrt{8}$ to $2\sqrt{2}$. So, this question is a classic example of one that could be in the non-calculator section [of the exam]. In fact, that could be a good heading for that example. [Writes the heading "Example of a non-calculator function unknown question"]

On completion of the problem a student, Tom, raised his hand and asked Mr Jones about an alternative solution method.

Tom: Mr Jones is there another way that question could be done?

Mr Jones: Tell me.

Tom: Well working out the perpendicular line, working out the intercept and then working out the distance using the distance formula.

Mr Jones: Um [pauses] OK say that again.

Tom: Working out the perpendicular line

Mr Jones: When you say working out the perpendicular line, what do you mean?

Tom: Yeah by using that point (1,1), and you use simultaneous equations to find that point in the middle. [Points to point Q shown on the diagram]

Mr Jones: OK, so I know what you are saying here, so you find out the gradient of that line. [Points to the line $y = x - 4$]

Tom: Yeah which you can by looking at it.

Mr Jones: Yeah and so if you find the gradient of that line and then worked out the gradient of the perpendicular line and used $y - y_1 = m(x - x_1)$ you could actually do it, yes you could [pauses]. That would give the equation of that line but would that give us the distance though?

Tom: Yeah because you could find it using simultaneous equations. Then you can use the distance formula to find it [the distance between points P and Q].

Mr Jones: Yes, yes you could, good point. Well done.

4.2.2 Mr Jones' Perspective

When asked whether he had used any different approaches or strategies in his teaching of applications of differential calculus this time around, Mr Jones offered the following response:

Mr Jones: Yes, I have, I've probably identified key words in the question and making sure that they understand what the process is. As I said, the difficulty of teaching calculus with maximums and minimums is that there can be a wide variety of questions, you know, finding maximum volume, maximum distance and all sorts of different things. So, when you are teaching a topic like that, and there are a few topics I teach like that in this course, you have to give them the principles behind what they are doing. So, these are the key steps you've got to do give them the framework I suppose and hopefully they can apply that framework to understanding other situations. So that's what I've been concentrating on more. I've done that in the past, but this year I've really concentrated on, and you might have heard me say some things today, "you know if it says minimum volume you need an equation for volume and find the minimum" and then there are the ones you have to get everything in terms of the one variable. I've concentrated on that a fair bit so yeah.

He also explained why he had chosen to focus on the "distance problem".

Mr. Jones: I wanted to give them an example of one that didn't require use of the calculator at all, because the nature of our course is that there is a calculator and a non-calculator section of the exam. So that was an important example because, I mean, I don't want to get too caught up in the exam, but in reality, I have to be faithful to anticipating what sort of questions come up.

When invited to comment on his own response to Tom's suggestion of an alternative solution, Mr Jones provided the following comments:

Mr Jones: Yeah well, I had to be honest in one sense that the pragmatic thing is that it [the "distance problem" in Figure 4.1] could be in the calculus section of the exam so you would have to use calculus. But what I liked about it [Tom's suggestion] was, I mean you don't want to shut kids down, you want them to explore what the alternative is so I tried not to, you know, say "Oh yeah good idea Tom" and then move on. I wanted him to explain how his alternative method could work. The exciting thing for me is that he recognized things that we've covered before with gradient and perpendicular lines and that sort of thing. So, he was taking prior knowledge, not just from this year but from previous years and seeing it in that context which was really exciting and you like the fact that kids are thinking, they are not just going with the flow they are actually actively thinking, you know how else could we do this?

4.2.3 The Students' Perspective

In their short reflections and focus-group interview, several students commented on the usefulness of Mr Jones' worked solution to his chosen optimisation problems including the "distance problem".

> When he did the steps on the board I could just look back to see how to do it. (Angela, focus-group interview response)

> The most useful thing were the problems done on the board to find minimum/maximum distance between points. (Harry, short reflection)

> His step-by-step examples were very useful for the harder question. (Danny, short reflection)

> The examples on the board were the most helpful because they helped me recognise when to do what e.g. when to look for min or max and making d'(x)= 0 (Lucy, short reflection)

> The explanations and worked examples on the board with consistent pausing to further explain items helped me gain an understanding of the work. (Keira, short reflection)

> The whiteboard examples were the most helpful. He [Mr Jones] engaged everyone in the class and you had to pay attention as he asked people for different values and numbers. (Christopher, short reflection)

Tom, who had originally asked about the alternative method of solution to the "distance problem", also commented that Mr Jones' explanations had been helpful for the differentiation of procedurally challenging expressions:

> The explanations were helpful for differentiating square roots with more than one thing in it, like when there was x squared plus 2x and then the square root of all that and you had to differentiate it. (Focus-group interview response)

The researcher invited Tom to comment further about his alternative solution to the minimum distance problem.

Tom: Yeah, I did it my way first [directly using the knowledge that the two lines are perpendicular] and got the right answer and then did it the other way [using calculus]. Well my version was quicker, but it can't be used like universally.

It would have been valuable to have asked Tom to elaborate on what he meant by “can’t be used like universally” but this information was not sought during the focus-group interview.

4.2.4 Commentary

Mr Jones’ worked solution to the “distance problem” was broadly classified as *teacher demonstration* and *concentration on procedures*. *Knowledge of examples* and *knowledge of assessment* were also evident when he discussed the suitability of the “distance problem” as a potential question on the non-technology section of the final examination. His emphasis on the keywords relevant to the solution process (e.g., minimum, maximum) and the use of funnelling questions to guide the students step-by-step through the differentiation of the distance function, were coded as *classroom techniques*. *Deconstructing mathematics into key components* was also evident in the way Mr Jones encouraged his students to recognise the critical processes involved in solving optimisation problems (e.g., find the derivative, set the derivative to zero, solve for “*x*”).

Mr Jones’ response to Tom’s suggestion of an alternative method of solution to the “distance problem” was classified as *responding to students’ ideas*. The teacher listened and responded positively to Tom’s suggestion but did not pursue it in any detail. As such, an absence of *mathematical structure and connections* as also evident because Tom’s suggestion presented an opportunity to address connections between the two methods of solution. Mr Jones appeared to lack some content knowledge or perhaps certain content knowledge did not come to mind “in the moment”, since calculus can actually be used to verify that the perpendicular distance is, indeed, the shortest.

Mr Jones' post-lesson interview also provided evidence of a combination of *concentration on procedures*, *knowledge of assessment*, and *knowledge of examples* when he explained that the "distance problem" was typical of the kind of question the students may encounter in the non-technology section of the external examination. His comments about his emphasis on keywords and the processes that are central to solving all optimisation problems supported the researcher's observations and were coded as *classroom technique* and *deconstructing mathematics into key components*. The interview also offered insight into Mr Jones' *responding to pupil ideas* when he justified his response to Tom's suggestion based on the expectation that students would need to use calculus to solve optimisation problems in an examination situation. Therefore, Mr Jones' justification for *responding to students' ideas* in this instance, was linked to *knowledge of assessment.*

The focus-group interview responses and short written reflections suggest that several students identified Mr Jones' *teacher demonstration* as useful for their learning when they highlighted his step-by-step explanation for solving the "distance problem". Other students noticed and valued Mr Jones' *classroom techniques*. For example, Lucy commented on the usefulness of Mr Jones' emphasis on the keywords in the problem to identify the steps required to solve it, and Christopher acknowledged the use of questioning to engage the class actively in the solution process.

4.3 Scenario 2: The "Particle Problem" with Mr Jones

Table 4.2

Information relating to Scenario 2

Date of Lesson	4th August 2014
Teacher	Mr Jones
Topic	Applications of differential calculus to rates of change
Rationale for selecting the data used to construct this scenario	Both the teacher and his students place particular value on mastering algebraic techniques that they perceive to be procedurally challenging.
Elements of PCK	<u>Concentration on procedures</u> <u>Foundation (KQ)</u> <u>Teacher demonstration</u> Transformation (KQ) <u>Knowledge of examples</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Classroom techniques</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Getting and maintaining student focus</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Mathematical structure and connections:</u> Content knowledge in a pedagogical context (Chick et al. framework) <u>Anticipation of complexity:</u> Connection dimension (KQ) Clearly PCK (Chick et al. framework)

This scenario focuses on part of a lesson involving the application of differential calculus to rates of change, during which the students were assigned several problems to solve from their text book. The following lesson excerpt depicts the way in which Mr Jones assisted his students to solve part b) of the particle problem shown in Figure 4.2.

<u>The "particle item":</u>

A particle moves in a straight line so that its displacement from a point O, at any time t is

$x = \sqrt{3t^2 + 4}$

Find:

a) The velocity as a function of time
b) The acceleration as a function of time
c) The velocity and acceleration when t=2

Figure 4.2. The "particle problem" (Hodgson et al., 2013, p. 386)

4.3.1 Lesson Excerpt

As the students worked through the textbook items they had been assigned, Mr Jones roamed the classroom monitoring their progress and responding to queries as they arose. Before long it became apparent that the majority of the students were having difficulty with part b) of the "particle problem". The students were aware from their previous lesson that velocity and acceleration are the first and second derivatives of a displacement function respectively, but the source of difficulty lay in the process of differentiating the velocity function $\frac{3t}{(3t^2+4)^{\frac{1}{2}}}$, obtained in part a) to yield the acceleration function needed for part b).

The students were familiar with the quotient rule from previous lessons, but in this case they were unsure how to manipulate the negative fractional powers to simplify the expression for the acceleration function. Mr Jones responded by instructing the students to leave their desks and to gather closely around the whiteboard. He then guided the class through the differentiation of the velocity function to obtain the acceleration function (see Figure 4.3) which involved applying the quotient rule.

$\frac{dx}{dt} = \frac{1}{2}(3t^2+4)^{-\frac{1}{2}}$ x $6t$ (velocity function)

$\frac{dx}{dt} = \frac{3t}{(3t^2+4)^{\frac{1}{2}}}$

$\frac{d^2x}{dt^2} = \frac{3(3t^2+4)^{\frac{1}{2}}}{[(3t^2+4)^{\frac{1}{2}}]^2} - \frac{3t\,(3t^2+4)^{-\frac{1}{2}} \text{ x } 3t}{[(3t^2+4)^{\frac{1}{2}}]^2}$ (acceleration function)

$= \frac{3(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)} - \frac{3t\,(3t^2+4)^{-\frac{1}{2}} \text{ x } 3t}{(3t^2+4)}$

$= \frac{3(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)} - \frac{9t^2(3t^2+4)^{-\frac{1}{2}}}{(3t^2+4)}$

$= 3(3t^2+4)^{-\frac{1}{2}} - 9t^2(3t^2+4)^{-\frac{3}{2}}$

$= 3(3t^2+4)^{-\frac{3}{2}}[(3t^2+4) - 3t^2]$ step 4

$= \frac{9t^2+12-9t^2}{(3t^2+4)^{\frac{3}{2}}}$

$= \frac{12}{(3t^2+4)^{\frac{3}{2}}}$

Figure 4.3. Mr Jones' worked solution to part b) of the particle problem

Mr Jones demonstrated an approach that involved factorising $3(3t^2+4)^{-\frac{1}{2}}$ - $9t^2(3t^2+4)^{-\frac{3}{2}}$ by "taking out" a common factor of 3 $(3t^2+4)^{-\frac{3}{2}}$. He emphasised the importance of "taking out the factor of $(3t^2+4)$ with the smallest power" which is, in this case, $(3t^2+4)^{-\frac{3}{2}}$. To illustrate this point, Mr Jones likened the process to factorising the simpler expression $2x^2 + 4x$, and pointed out that the highest common factor $2x$ includes the common pronumeral with the lowest power. He then unravelled how 3 $(3t^2+4)^{-\frac{3}{2}}[(3t^2+4)\ \ 3t^2]$ was obtained, by prompting the students to recognise that $(3t^2 + 4)^{-\frac{3}{2}}$ x $(3t^2 + 4)^1$ is equal to $(3t^2 + 4)^{-\frac{3}{2}+1}$ and hence $(3t^2 + 4)^{-\frac{1}{2}}$. Mr Jones then guided the students through the rest of the simplification process as shown in Figure 4.3.

After the students had returned to their desks, one of the students, Alan, remained at the front of the classroom to finish copying down Mr Jones' solution.

4.3.2 Mr Jones' Perspective

The post-lesson teacher interview shed further light on Mr Jones' instructional decisions in relation to the "particle problem".

Mr Jones: That coming up to the board thing I've done that a bit this year and last year, where I just feel like if there's a really complex thing I'm explaining I just think the dynamic of being down the back and you know to be right up here where I can eyeball people just to make sure they are really engaged.

The negative index which was a fraction, the $-\frac{1}{2}$ and having that and saying, "Oh my goodness how do I get a common factor?" Well that's why I related it back to the $2x^2 + 4x$. "What do we do here? Ok the power that we use as the common factor is the lowest power". That's what made it so difficult, the negative index of a complex expression to start with, and then having to extract out of there a common factor.

4.3.3 The Students' Perspective

During the student focus-group interview, the researcher asked Alan about his experience with the "particle problem":

Alan: […] I wasn't really sure how to do the negative indices. Like having negatives and a fraction at the same time kind of threw me.

Researcher: What helped you in the end?

Alan: Just his [Mr Jones] explanation that the difference between $-\frac{1}{2}$ and $-\frac{3}{2}$ is just one, so just using that. Yeah and with the weird indices when he [Mr Jones] went through it slowly, he sort of went step by step and it was very easy to understand.

Of the 15 students who contributed written responses at the end of the lesson, 11 identified Mr Jones' explanation of part b) of the "particle problem" as the most useful aspect of the lesson. For example, Tom recorded that:

> The most helpful thing was when he showed us how to find the derivative of $3t\ (3t^2 + 4)^{-\frac{1}{2}}$ with the long explanation. It showed me how to take out common factors with powers of $-\frac{1}{2}$ and $-\frac{3}{2}$.

Dylan's response also focused on the way in which Mr Jones' had "helped us understand the$-\frac{1}{2}$ and $-\frac{3}{2}$ ". Other more general comments included "the most helpful thing was when we all went up to the front of the classroom and he talked us step by step through the problem" (Lucy).

4.3.4 Commentary

Mr Jones' worked solution to part b) of the "particle problem" was classified as *teacher demonstration*. His decision to gather the students around the whiteboard to engage them in the manipulation of the "tricky" expression was coded as both *classroom techniques* and *anticipation of complexity*. Evidence of *mathematical structure and connections* and *knowledge of examples* were also demonstrated when Mr Jones explained that he had chosen an arguably simpler expression (i.e., $2x^2 + 4x$) to draw attention to the idea that fractional indices behave the same as whole number indices.

During his post-lesson interview Mr Jones elaborated on his decision to gather the students around the whiteboard by claiming that he used this technique to engage the students' concentration when "there's a really complex thing" to explain. This response was classified as *concentration on procedures*, *anticipation of complexity* and *getting and maintaining student focus*. There is a sense in which this combination of elements of PCK suggests that Mr Jones prioritised procedural mastery of arguably challenging techniques. Evidence of *mathematical structure and connections* and *knowledge of examples*, which supported the researcher's lesson observations, were demonstrated when Mr Jones discussed his use of the simpler example, $2x^2 + 4x$, to demonstrate how an expression involving negative fractional indices can be factorised using the same principle.

Responses from the written reflections and focus-group interview suggest that the students appreciated Mr Jones' step-by-step explanation of how to simplify the "hard" expression, and these were coded as *teacher demonstration*. Some students, including Alan, particularly identified that learning how to "take out common factors with powers of $-\frac{1}{2}$ and $-\frac{3}{2}$" was the most useful aspect of the lesson.

4.4 Scenario 3: The "Particle Problem" with Mr McLaren

Table 4.3

Information relating to Scenario 3

Date of Lesson	5^{th} August 2015
Teacher	Mr McLaren
Topic	Applications of differential calculus to rates of change
Rationale for selecting the data used to construct this scenario	Teacher raises a student's awareness of the power of recognising flexible and efficient strategies to simplify expressions.
Elements of PCK	Teacher demonstration Transformation dimension (KQ) Overt display of subject matter knowledge Foundation dimension (KQ) Knowledge of examples Transformation (KQ) Clearly PCK (Chick et al. framework) Deconstructing mathematics into key components: Content knowledge in a pedagogical context (Chick et al. framework) Anticipation of complexity Connections (KQ) Clearly PCK (Chick et al. framework)

This scenario also features the "particle problem" (presented again as Figure 4.4), this time within the context of a lesson taught by Mr McLaren. It was not surprising that the researcher occasionally observed the same examples demonstrated by different teachers given that all students who studied Mathematics Methods (MTM315) used the same prescribed text-book.

The "particle item": A particle moves in a straight line so that its displacement from a point O, at any time t is $x = \sqrt{3t^2 + 4}$ Find: a)The velocity as a function of time b)The acceleration as a function of time c)The velocity and acceleration when t=2

Figure 4.4. The "particle problem" (Hodgson et al., 2013, p. 386)

4.4.1 Lesson Excerpt

Mr McLaren's lesson, like Mr Jones' lesson featured in the previous scenario, focused on applications of differential calculus to solving problems involving instantaneous rates of change. During his lesson Mr McLaren assigned several items, including the "particle problem" for his class to complete. At the request of three students, the teacher modelled the solution to part b) of the problem on the white board. These students, like those in Mr Jones' class, had been experiencing difficulty manipulating the second derivative. Mr McLaren's method of solution is shown in Figure 4.5. It is also worth noting that his students had been previously introduced to, and had used, the quotient rule to differentiate rational functions.

$x = \sqrt{3t^2 + 4}$ (displacement function)

$= (3t^2 + 4)^{\frac{1}{2}}$

Part a) $\frac{dx}{dt} = \frac{1}{2}\,(3t^2 + 4)^{-\frac{1}{2}}$ x $6t$ (velocity function)

$\frac{dx}{dt} = \frac{3t}{(3t^2+4)^{\frac{1}{2}}}$

$\frac{d^2x}{dt^2} = \frac{[3(3t^2+4)^{\frac{1}{2}} - 3t\,(3t^2+4)^{-\frac{1}{2}} \times 3t]}{((3t^2+4)^{\frac{1}{2}})^2}$ x $\frac{(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)^{\frac{1}{2}}}$

$\frac{d^2x}{dt^2} = \frac{3(3t^2+4) - 9t^2}{(3t^2+4)^{\frac{3}{2}}}$

$\frac{d^2x}{dt^2} = \frac{12}{(3t^2+4)^{\frac{3}{2}}}$ (acceleration function)

Figure 4.5. Mr McLaren’s process for simplifying the second derivative in part b) of “the particle problem”

Mr McLaren simplified the expression for the acceleration function by multiplying it by $\frac{(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)^{\frac{1}{2}}}$ as shown in Figure 3.5. When the final answer of $\frac{12}{(3t^2+4)^{\frac{3}{2}}}$ was obtained Grace, one of the students, remarked with a smile “all that work for just one silly particle”. Carl asked Mr McLaren “where did the $(3t^2 + 4)^{\frac{1}{2}}$ over $(3t^2 + 4)^{\frac{1}{2}}$ come from?” Mr McLaren explained that it is “a useful technique to enable us to simplify the expression. You are not changing anything you are just multiplying by one.”

4.4.2 Mr McLaren’s Perspective

During his post-lesson interview, Mr McLaren discussed his approach to addressing the “particle problem”:

Mr McLaren: The common difficulties are like that example with the acceleration [the “particle problem”] where they find that once they’ve used the quotient rule, in particular, if there are interesting parts to that expression like having a fraction index in the numerator they don’t always know how to simplify that and fractions in general students have trouble with**.** I would say with that particular question students have a problem with it every year because it has a bit of everything in it. It has the second rate of change so acceleration from velocity and then using the quotient rule. Then when they look in the answers at the back of the book – because that's what they do – and it looks way different to what they've got so they say, how can you get something simple from something like what I've got which looks so complex.

4.4.3 The Students' Perspective

In the focus-group interview, Carl made the following comment:

Carl: When he [Mr McLaren] had the equation up there and he wrote something to the power of $\frac{1}{2}$ over the exact same thing and it cancelled out, I thought that was cool because it equalled one. I was stuck with that question, thinking what am I doing, how am I going to get rid of all this? But then he's like 'no you just kind of like look outside the box and just think about it, if you write that [he is referring to $\frac{(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)^{\frac{1}{2}}}$] then it's 1, but it lets you cancel other stuff out.

Carl also provided the following written response: "I learnt to look outside the box when trying to simplify something". Similarly, Grace and David offered responses that focused on Mr McLaren's worked solution to the "particle problem":

David: The most helpful thing was his [Mr McLaren's] explanation of calculating velocity and acceleration and where it had a square root and you had to use the quotient rule. (Focus-group interview)

Grace: It [the "particle problem"] had many of the different concepts all in the one question and it helped me apply all that. (Short reflection)

4.4.4 Commentary

Mr McLaren's worked solution to part b) of the "particle problem" was coded as *teacher demonstration*. His approach to simplifying the expression for the second derivative was classified as *overt display of subject matter knowledge* because he appeared to select a particularly efficient method of simplifying the expression. Mr McLaren was not particularly didactic in his approach, but quietly provided a worked solution on the board at the request of three students. His response to Carl's question about where the "$(3t^2 + 4)^{\frac{1}{2}}$ over $(3t^2 + 4)^{\frac{1}{2}}$" came from was classified as *deconstructing mathematics into key components.* That is, Mr McLaren unpacked his own content knowledge to make the simplification technique explicit to Carl.

Mr McLaren's post-lesson interview comments were coded as both *anticipation of complexity* and *knowledge of examples*. In particular he highlighted the features of the problem that impact on its complexity, as well as the typical ways in which students respond to the challenges of "the particle problem" based on his previous teaching experiences.

Data from both the focus-group interview and the short reflections suggest that several students particularly appreciated Mr McLaren's demonstration of how to simplify the expression for the second derivative. David's comment about the usefulness of Mr McLaren's explanation was aligned to *teacher demonstration.* Grace's response was linked to *knowledge of examples* since she alluded to similar features of the "particle problem" identified by Mr McLaren. Carl's comment was identified as *mathematical structure and connections* because he claimed to have made some generalizable connections in relation to simplifying difficult expressions as a result of Mr McLaren's explanation.

4.5 Scenario 4: Anti-Differentiation with Mr Taylor

Table 4.4

Information relating to Scenario 4

Date of Lesson	19th August 2014
Teacher	Mr Taylor
Topic	An introduction to the process of anti-differentiation
Rationale for selecting the data used to construct this scenario	Teacher focuses on the connection between the processes of anti-differentiation and differentiation, an approach that is appreciated by the students.
Elements of PCK	<u>Theoretical underpinning of the pedagogy:</u> Foundation (KQ) <u>Adherence to textbooks</u> Foundation (KQ) <u>Use of mathematical terminology</u> Foundation (KQ) <u>Knowledge of curriculum</u> Clearly PCK (PCK framework) <u>Teacher demonstration</u> Transformation (KQ) <u>Knowledge of examples</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Classroom technique</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Deconstructing mathematics into key components</u> Content knowledge in a pedagogy context (Chick et al. framework) <u>Knowledge of pupil errors</u> Foundation (KQ) <u>Mathematical structure and connections</u> Content knowledge in a pedagogy context (Chick et al. framework)

This scenario focuses on a lesson involving an introduction to the process of anti-differentiation. The class had previously completed a unit of work on differential calculus as part of the Mathematics Methods course.

4.5.1 Lesson Excerpt

When the researcher arrived to observe the lesson, Mr Taylor was searching through some notes he had brought to class: "I seem to have lost the first page of my notes. Ok so we'll have to go without notes." Without delay he went to the whiteboard and the lesson commenced.

Mr Taylor: We need to start looking at the topic of integration. Now the calculus that we started with was the calculus of differentiation. So, if I have some function of *x* and I take its derivative with respect to *x* then that gives me *f* prime of *x*. Now if we look at a function [writes $\frac{d}{dx}(3x-8)=3$ on the whiteboard].

Mr Taylor: Now we have learnt a number of rules to help us out with differentiation. We learnt the power rule, the product rule, the quotient rule, the chain rule, and most of us did the test and did very well. So, with differentiation we start off with a function and when we differentiate it, it gives us the gradient function. So now we've got to do something that starts off with the gradient function and ends up with the function. I want to go back and **undo** [teacher places emphasis on the word 'undo'] what I did before. We want to start with the gradient function and end up with the function [scribbles this information on the whiteboard]. So, if the derivative of $3x$-8 equals 3 then we want to say if I start with 3 I want to end up with $3x$-8, I want to go back and undo what I did before. Integration at the level we are doing it is often called anti-differentiation. Now putting it in more general terms, I want the integral of "*f* prime of *x*" with respect to *x* and that's going to give us *f(x)*. This symbol here is read as 'integral' [points to the integral sign as shown in Figure 4.6] and it is simply an 'S' stretched down.

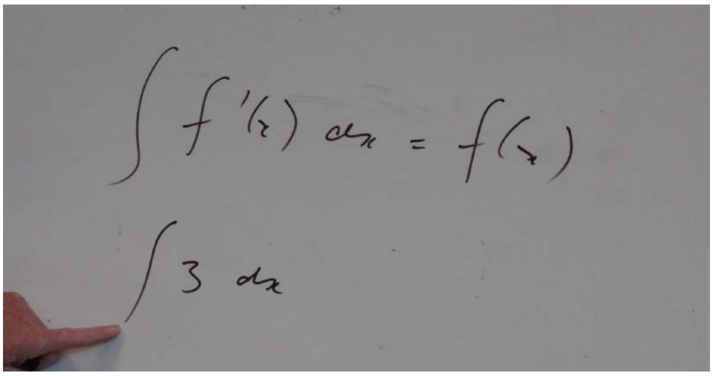


Figure 4.6. Mr Taylor introduces the integral symbol

Mr Taylor: Because one of the ways we get the integral is by summing the function. It's originally an 'S' for a sum and we are stretching it out to make it into the integral sign. So, if I have the integral of 3 with respect to *x*, the operator with respect to x [Points to dx in $\int 3\ dx$] must go along with the integral symbol, they go as a pair. You can't use one without the other. So, we have got the integral of 3 with respect to *x* is equal to 3*x*.

The instructional phase of the lesson continued with the introduction of the arbitrary constant of integration.

Mr Taylor: But we had minus eight in there? I wonder what would happen if instead of having minus eight I took the derivate of $3x + 12$. What's its derivative going to be? It's still 3. So looking back over here we said that the integral of 3 was 3*x* [points to $\int 3dx = 3x$], over here we said the integral of 3 was $3x - 3$ [points to $\frac{d}{dx}(3x-8) = 3$], and over here we said the integral of 3 was 3x + 12 [points to $\frac{d}{dx}(3x + 12) = 3$]. It looks a bit chancy doesn't it, it could be anything. The 3x stays the same, but the other term could have been 8, could have been zero, could have been 12, it could have been what? Any? [one student says, "It could be anything." and Mr Taylor probes for a more specific response] Not anything but, a simple word beginning with c that means a number with a precise value. [Margot replies "Any constant."] Yes, good, it can be any constant. So, when we are integrating a single term like that we need to put in the plus c, the constant [Mr Taylor adds the constant c as shown in Figure 4.7].

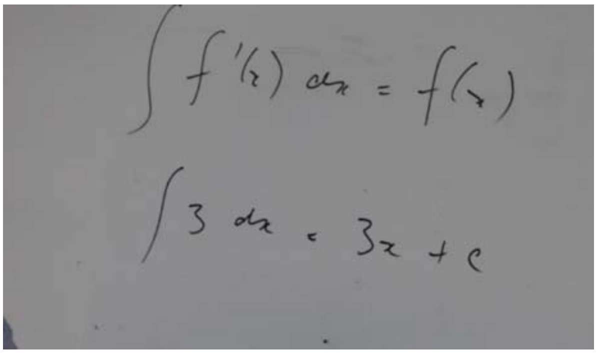


Figure 4.7. Mr Taylor addresses the need for the arbitrary constant of integration

Mr Taylor: When we integrate in a manner like this we have to have a plus c because of some constant there. It is called "the arbitrary constant." [Writes the term on the board]

This was followed by an explanation of how the arbitrary constant should be used to align with the expectations of text-book and examination questions.

Mr Taylor: If the question asks to find ***the*** anti-derivative then the plus c must be used. But if the question says find ***an*** anti-derivative then we do not use plus *c*. What we are doing is setting *c* as zero. Examiners and text books will expect that if the question says to find the anti-derivative they will expect to see the plus c stuck on the end. If it says find ***an*** anti-derivative then anyone will do and the one which has *c* equal to zero, and therefore is the easiest one to write.

Following the discussion of the arbitrary constant of integration, Mr Taylor introduced the process of anti-differentiating (with respect to *x*) expressions of the form ax^n where *n* does not equal negative one. He began by using the example $\frac{d}{dx}3x^2 = 6x$ to review the steps involved in differentiating expressions of the form ax^n with respect to *x*, to illuminate the connection between the processes of differentiation and anti-differentiation:

Mr Taylor: How do I go about getting the derivative of a term like $3x^2$ Michelle? [Michelle responds that the 3 is multiplied by the index and the index becomes 1] Yes that's right I take that power of 2 and make it a factor which gives us 2 x 3 x x. So, you did two things didn't you, you took the index and made it a factor. And what was the other thing you did? Yes, the index went down by one [He annotates the steps on the whiteboard as shown in Figure 3.8]. So, if I'm going to get the integral of $6x$ with respect to x, what am I going to have to do? I'm going to have to undo those two steps.

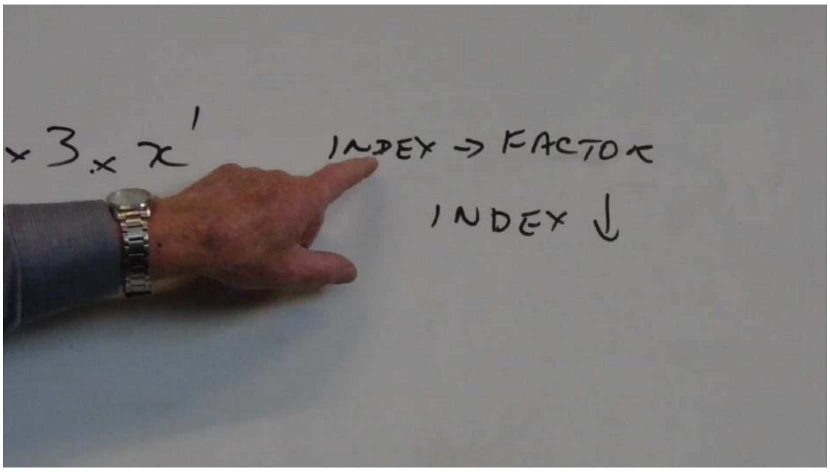


Figure 4.8. Mr Taylor annotates the steps involved in differentiating expressions

Mr Taylor then encouraged the students to recognise that undoing “those two steps” to find the integral of 6*x* involved “increasing the index by one” and then “dividing by the new index”. He emphasised these steps by listing them on the white board as shown in Figure 4.9.

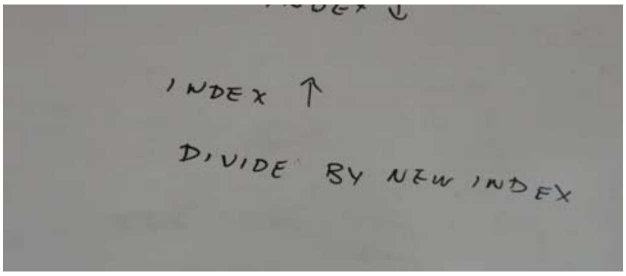


Figure 4.9. Mr Taylor annotates the steps involved in anti-differentiating expressions

Mr Taylor then led the class through the solution to each of the examples given in Figure 4.10, in the order shown.

1. $\int x^7 dx$

2. $\int 4x^{-3} dx$

3. $\int -6x^{-4} dx$

4. $\int \frac{5x^{-4}}{3} dx$

5. $\int x^{\frac{-3}{7}} dx$

Figure 4.10. Sequence of examples involving anti-differentiating expressions

After the class had practiced differentiating expressions of the form ax^n (with n ≠ -1) Mr Taylor wrote $\int 5\ x^{-1}$dx on the whiteboard. Initially, he did not draw attention to the fact that this example was going to be different, and nominated one of the students to begin anti-differentiating $5\ x^{-1}$ with respect to *x*.

Mr Taylor: Ok just watching up at the board while Jake works out the next question for us please. Jake, we want the integral of 5 times *x* to the negative one with respect to *x*. Now the steps we have learnt so far are to increase the index by one and divide by the new index. Ok, let's go – will we leave the 5 where it is, happy with that? What about the *x*?

Jake: Well that's going to go to zero isn't it?

Mr Taylor: And the next step is?

Jake: [Hesitates for a couple of seconds] Plus *c*?

Mr Taylor: No.

Jake: [Looks puzzled and uncomfortable] Um divide by zero?

Mr Taylor: [Writes the steps on the board on shown in Figure 4.11] Divide by zero and then plus *c*. What do you know about that?

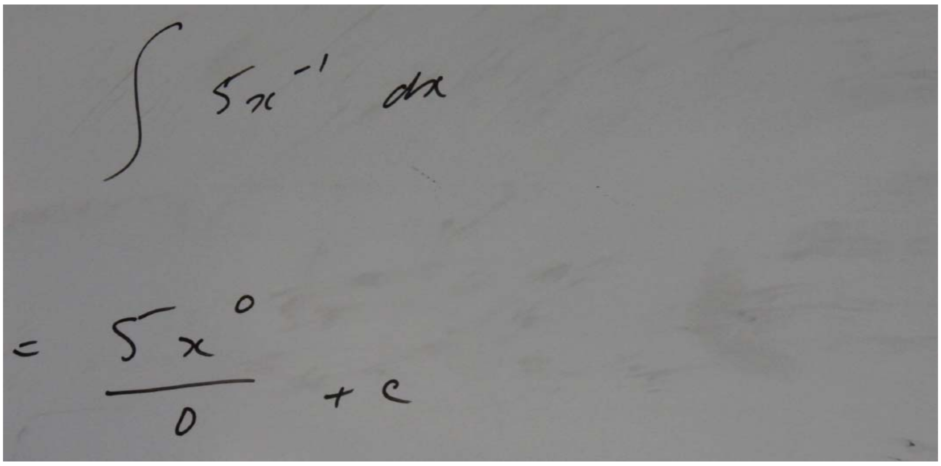


Figure 4.11. Mr Taylor incorrectly divides by zero to deliberately make a point

Jake: Well it's undefined isn't it?

Mr Taylor: It's undefined. It doesn't exist, so we can't use it. All the ones [anti-differentiation examples] we have done so far are on the proviso that the index we start off with is not negative one. We need a separate rule altogether when it is negative one. So, all the procedures we've been doing like adding one to the index and dividing by the new index, that's only if n is not equal to negative one. What about if it is equal to negative one? [He clears the board]. We have done this before [Writes the derivative with respect to x of the natural logarithm function ln x on the board]. What's the derivative of ln x?

Jake: It's 1 over x.

Mr Taylor: Yes, it's 1 over x. Therefore, the integral of 1 over x with respect to x should be, going backwards, gives us ln x. But wait a minute, to say that ln x exists, what have I got to do?

Jake: Make it not equal to a negative.

Mr Taylor: Yes, so how can I make it so it's not equal to a negative?

Jake: Make it equal to a positive

Mr Taylor: Make it equal to the absolute value of x. [See Figure 4.12]

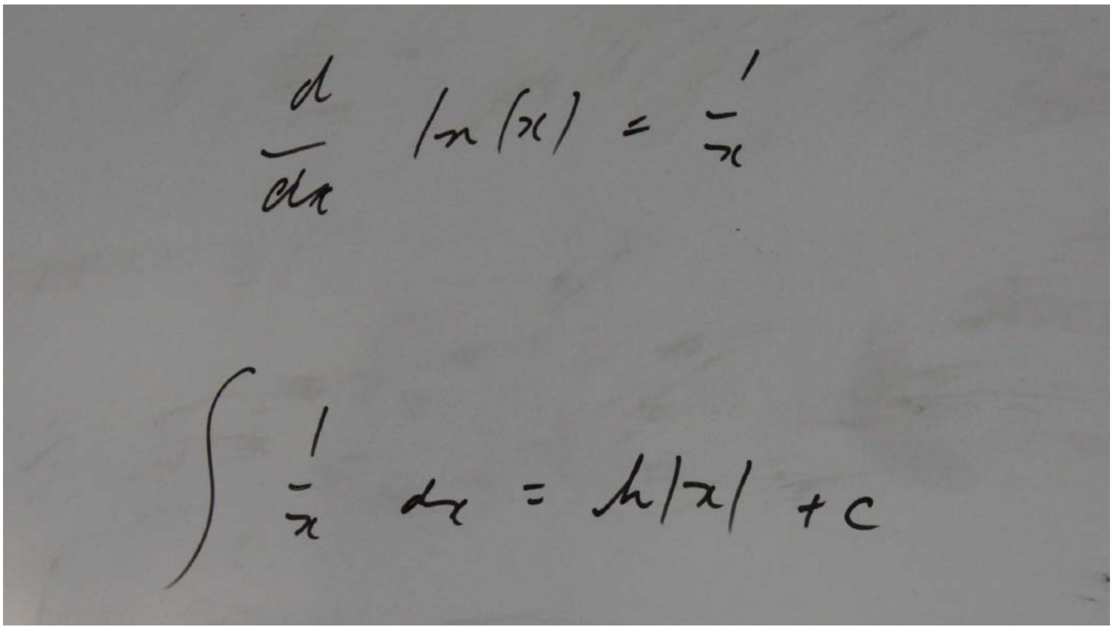


Figure 4.12. Mr Taylor introduces the idea of anti-differentiating functions

The original example $\int 5\, x^{-1}\mathrm{d}x$ was then resolved to $5\ln|x| + c$.

4.5.2 Mr Taylor's Perspective

In his post-lesson interview Mr Taylor highlighted some of the challenges that students tend to experience when learning the processes of anti-differentiation:

Mr Taylor: I first taught this topic in 1977, I think. I think the advantage of teaching it a large number of times, is that you get to know where kids are likely to run into trouble and when they are going to make an error, and you can see in advance what kinds of errors they are going to make. The main trouble is not sorting out the differences between differentiation and integration – that's the biggest difficulty with people who are struggling a little bit and don't quite know how to integrate so they try and fall back on their differentiation techniques. If they don't get the two processes sorted out realising they are opposites and therefore can't be switched – so they'll add one to the index but then multiply by the new index.

He also discussed his choice and sequence of examples:

Mr Taylor: I started off with whole numbers and then started getting involved in negative numbers, particularly negative indices, and then into fractional ones. That's where the kids have difficulty, switching to working with a negative number so they do negative three plus one is negative four. Each one [example] has a reason for being there. This one's got this, this one's got that. They may just be slight differences but that little bit makes a very big difference.

The researcher invited Mr Taylor to comment on his emphasis on the correct use of integral notation and terminology:

Mr Taylor: Well the examiners' comments every year say something like 'students do not appreciate the operator '*dx*'. They [the students] either, leave it out, or in some cases put it in silly places. One year the examiners' comments pointed out that a number of students had written the *dx* operator directly inside the integral sign, which showed they perhaps had been drummed up on showing both symbols but hadn't understood what we're talking about.

During the interview, the researcher mentioned how the students had expressed appreciation for the way in which Mr Taylor had introduced the sequence of steps involved in anti-differentiation of functions by labelling the steps involved.

Mr Taylor: Yes, if you can leave those instructions on the board while they are working on the next problem then they'll use the instructions quite well. I often try to say to the kids that I do a lot of talking during lesson time, but the talking is not so much direct instruction it's this is what should be going through your head already. I'll go through those steps I'll go through them again and again until you get so sick of hearing them they are embedded in your head. So, trying to get inside their heads, I don't know how successful it is sometimes, but we keep on trying.

During his interview Mr Taylor also discussed his approach to introducing the students to the integral of $\int 5x^{-1}dx$ and reflected on the ways in which he had adapted his teaching practice based on past experiences.

Mr Taylor: I wanted the class to be following and I wanted them to realise that something was wrong. Jake, picked it up quite quickly but he is a very quiet speaker and he didn't call it out loud enough for everyone else to hear the fact that the power of negative one was a problem when you added one to it. I was actually trying most of the way through the lesson to show them that this is something that is not new. They've learnt how to differentiate, so really, they are just taking what they've learnt before and then using it backwards. What was different about that last one [$\int 5\ x^{-1}\ dx$] was the initial rules about adding one to the index and then dividing by the new index no longer applied but the process of doing the opposite to differentiation still applied. I've tried this year to be a little bit less theoretical. I mean rather than giving them a rule and saying, "follow that rule all the time", I've done more of "let's take it through step by step and see what we are doing here". Last year I did a bit more of "here's the rule, here are the steps, I'll do one example for you, now you go on and do it". But then I found that there was this big group of boys over in the corner [points to an area of the classroom] and they did absolutely nothing, they switched off because they couldn't follow the rule and they twiddled around and did it on the calculator or something. The girls on the other hand who sat on the other side of the room who did a lot of questioning, talking and bringing things out all got through. The boys weren't willing to recognise that, well, if it's on the board it's something they need to know, they'd sit back and wait to be told what they were supposed to do, whereas the girls came out and asked the questions. So, I've done more this year to say "what are the steps we are going through" rather than me just telling them because if I tell them they don't seem to listen. In the one that didn't work for example [he is referring to the fact that the power rule "doesn't work" for $\int 5\, x^{-1}\, dx$] I tried to make it so it was let's work through it and see what happens, rather than saying this one doesn't work and write down the rule.

Mr Taylor's interview took place after his final lesson involving indefinite integrals. During the interview he contemplated how he might introduce definite integrals and the application of integration to calculating areas, in his next lesson.

Researcher: Have you done anything different with your teaching of integral calculus this year compared to previous years?

Mr Taylor: Not really. I've got to decide whether I'm going to do the trapezoid rule or not bother and go straight to the definition from the fundamental theorem. [The trapezoid rule is a numerical method of approximating the area enclosed by a function].

Researcher: How will you decide which way to go?

Mr Taylor: Well it comes back to a matter of saying that for a new concept it would mean about a period and a half. Have I got a period and a half to spare? Probably not, so I'll probably leave it out on the grounds that it's not specifically mentioned in the Tasmanian syllabus. It is in the Victorian syllabus though.

Researcher: If you had it your way, so to speak, in that there were no limitations with the syllabus, would you teach the trapezoid rule?

Mr Taylor: Yes, because it makes clear, or clearer, to the kids as to why they've got to end up with a smooth curve and hence integrate. So, if we put in 5 columns and work out the area and then cut all the columns in half and have 10 columns taking up the same space what's going to happen? It's going to start approximating a curve – if we have 200 columns what's going to happen and so on. Or if we get to the stage where you can put as many columns as you could possibly imagine, and they are as thin as you can possibly imagine what are we going to be getting – Oh yes we're colouring in the whole area.

4.5.3 The Students' Perspective

In their post-lesson focus-group interview several students commented that they appreciated the explicit and stepwise approach to anti-differentiation provided by Mr Taylor, as suggested in the following excerpt:

Jake: Well basically integrating is the opposite of differentiating. So, you do the steps in the opposite order sort of thing.

Researcher: And what in particular helped you to get to know that?

Jake: The way Mr Taylor explained on the board, like wrote out the steps.

Victoria: Well he listed them for us [Giggles earnestly as my question seems so obvious?]

Liam: He annotated what you actually have to do.

Margot: I think last year when we were doing it a little bit I just kind of thought, like just tried to do it all backwards. But now having all the steps, you have an order to follow rather than trying to think about everything backwards I guess.

Vincent: Yeah showing a set method that you can work through and that applies to every question really helps.

Liam: I guess using diagrams and writing it out is easier for us to visualise rather than him just doing it and saying it because sometimes I forget it like I'm so busy writing it down I forget it.

Vincent: Yeah you can go back to your notes with that written method and see what is happening because generally it's a guessing game as to what happened because you're too busy writing down the notes.

Tom: Yeah, you're not actually taking in what he's saying and then when you look back [at your notes] you think oh I don't really know how to do that.

4.5.4 Commentary

In spite of the absence of his planning notes, Mr Taylor introduced the topic of integration with fluency and made explicit links to the process of differentiation. This aspect of the lesson was broadly coded as *teacher demonstration* as well as *mathematical structure and connections* given the teacher's emphasis on the connection between differentiation and anti-differentiation. Mr Taylor's explicit focus on the appropriate use of the operators ∫ and *dx*, were identified as *use of mathematical terminology.* His emphasis on how the prescribed textbook distinguished the constant of integration '*c*' depending on whether the problem asks for "an" antiderivative or "the" antiderivative was coded as *adherence to textbooks.*

The teacher's choice and sequencing of examples which involved a range of integrals of the form ax^n (where $n \neq -1$) was aligned to *knowledge of examples.* Mr Taylor's strategy of invoking cognitive conflict by allowing the students to initially approach $\int 5\, x^{-1}\, dx$ by "increasing the power by 1" was coded as *classroom technique* as well as *knowledge of examples.*

Mr Taylor's post-lesson interview comments about the common errors that students tend to make in relation to processes of anti-differentiation were coded as *knowledge of student errors*. He also attributed this knowledge to his many years of experience teaching integral calculus. As part of the interview Mr Taylor explained that his emphasis on the correct use of the operators, $\int$ and dx was based on the advice provided in "examiner's comments". Examiners' comments are part of an overall "Examiners' report" which is compiled after the marking process each year by those involved in setting and or marking the examination paper. As such, Mr Taylor's *use of mathematical terminology* (and notation) was linked to his knowledge of the Mathematics Methods syllabus documentation, hence *knowledge of curriculum*.

During his interview, Mr Taylor also highlighted the idea of "taking what they've [the students] learnt before [involving differentiation] and then using it backwards". This comment was coded as *mathematical structure and connections* in line with the researcher's observations of the way in which Mr Taylor drew explicit connections between the processes of differentiation and anti-differentiation. He also explained that his demonstrations were intended to model the kind of thinking that "should be going through your [the students'] head already" rather than "direct instruction". In addition, his comments suggested that students are more likely to learn by exploring how a particular process works, or does not work as the case may be, rather than simply being told from the outset. These reflections on the effectiveness of the ways in which students engage with their own mathematics learning were classified as *theoretical underpinnings of the pedagogy*.

The students' interview responses suggest that they appreciated how Mr Taylor articulated the steps involved in anti-differentiating expressions of the form ax^n ($n \neq -1$) and were coded as *teacher demonstration*. Jake's first interview comment was classified as *mathematical structure and connections* because it focused on the connection between the process of anti-differentiation and differentiation. The other students who participated in the focus-group interview focused particularly on Mr Taylor's strategy of annotating the examples (*classroom techniques*) to highlight the order of steps required and their connection to the process of differentiation (*mathematical structure and connections*). Several students also highlighted the usefulness of Mr Taylor's annotated examples by suggesting they would be able to interpret the solution process more easily when they looked back at their notes.

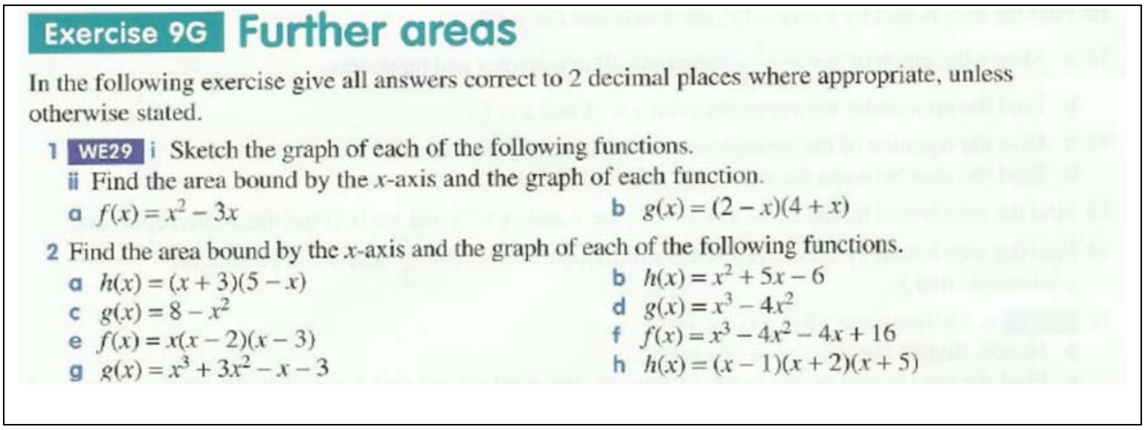
4.6 Scenario 5: "It's not a perfect square"

Table 4.5

Information relating to Scenario 5

Date of Lesson	1st September 2014
Teacher	Mr Jones
Topic	Applications of integral calculus to finding areas enclosed by functions
Rationale for selecting the data used to construct this scenario	Teacher helps his students to recognise that a particular "rule" may be generalised beyond familiar cases.
Elements of PCK	<u>Theoretical underpinning of the pedagogy</u> Foundation dimension (KQ) <u>Knowledge of examples</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Mathematical structure and connections</u> Content knowledge in a pedagogical context (Chick et al. framework) <u>Anticipation of complexity</u> Connections (KQ) Clearly PCK (Chick et al. framework)

This scenario is based on part of an integral calculus lesson during which the students were assigned an exercise (see Figure 4.13) that involved sketching graphs and finding areas enclosed by functions.



Exercise 9G Further areas

In the following exercise give all answers correct to 2 decimal places where appropriate, unless otherwise stated.

1 WE29 i Sketch the graph of each of the following functions.
ii Find the area bound by the x-axis and the graph of each function.
a $f(x) = x^2 - 3x$ b $g(x) = (2 - x)(4 + x)$

2 Find the area bound by the x-axis and the graph of each of the following functions.
a $h(x) = (x + 3)(5 - x)$ b $h(x) = x^2 + 5x - 6$
c $g(x) = 8 - x^2$ d $g(x) = x^3 - 4x^2$
e $f(x) = x(x - 2)(x - 3)$ f $f(x) = x^3 - 4x^2 - 4x + 16$
g $g(x) = x^3 + 3x^2 - x - 3$ h $h(x) = (x - 1)(x + 2)(x + 5)$

Figure 4.13. Exercise from the prescribed text (Hodgson et al., 2013, p. 435)

4.6.1 The Lesson Excerpt

The class had been working through the text-book exercise, and before long several students raised their hands to ask for help with a question that required them to find the area bound by the x-axis and the graph of $g(x) = (8 - x^2)$. Mr Jones called the whole class to attention, wrote the expression $8 - x^2$ on the white board, and asked "How does it factorise?" One of the students, Simon, responded with "Plus four and minus four". The teacher respectfully acknowledged the incorrect response and asked for other suggestions. There was some mumbling, and an audible "I dunno", as the students appeared to ponder the question. Mr Jones paused for a short time and then said, "I'll do a different one". He wrote $9 - x^2$ on the white board and prompted the students to recall that the expression is a difference of two squares, "How does it factorise, Mark?"

Mark hesitated a little before providing the correct response, "Three plus x, three minus x". Mr Jones nodded and pointed to the factorised expression (see Figure 4.14) and asked, "What will our x intercepts be for that graph?" Mark mumbled "Um, x equals negative three and x equals positive three?"

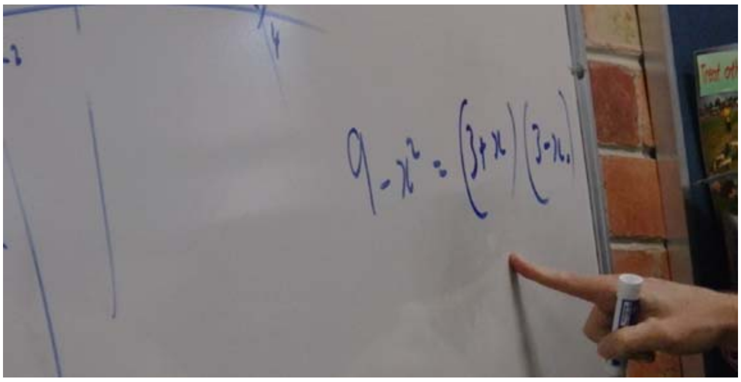


Figure 4.14. Mr Jones chooses a familiar example to prompt students' thinking

Mr Jones then returned to the $8 - x^2$ example and explained "This isn't like one of our perfect squares like 16 or 25 or 36 and such. But you can still apply that difference of perfect squares principle or rule to it, tell me how?" There was a short pause before James correctly answered, "The square root of 8 minus *x* and the square root of 8 plus *x*". Mr Jones nodded enthusiastically and factorized the expression accordingly. "Often people tune in and look for 9 and 25 and 36 and so on. But that difference of perfect squares can be used for any number" Mr Jones explained. He then pointed out that "We really should convert this [square root of 8] to two root two". Mr Jones wrote the equivalent form on the board and asked the class where the *x* intercepts will be. Alan correctly answered, "Positive two root two and negative two root two".

After the students had resumed working through their assigned questions, Mr Jones quietly remarked to the observing researcher:

Mr Jones: You probably found this when you were teaching, there is a tension in class, there is that temptation to jump up to the board and say here's how you do it. I don't know there's always that tension between troubleshooting and knowing what's going to trip them up, but you don't want anything to hold them up and you get in and you try to troubleshoot for them.

4.6.2 Mr Jones' Perspective

While discussing the lesson during his interview, Mr Jones reiterated the tension between presenting students with situations that are "likely to trip them up" and "the temptation to jump up to the board and say 'here's how you do it'":

Mr Jones: I stopped the class for the $8 - x^2$ one and I may have jumped the gun a bit too much, but I saw a few people stuck on it going 'how am I going to do this'.

He also highlighted that students tend not to recognise straight away that expressions such as $8 - x^2$ may be expressed as a "difference-of-two-squares":

Mr Jones: This is something they consistently do – they don't recognise perfect squares outside the situation where they've got two perfect squares like 9. If they see 8 they don't register 8 as a perfect square [sic, not technically a perfect square] and hence they don't think about going on and saying, "Oh yeah it's the square-root-of-8-squared minus *x* squared" So I deliberately chose that one just to get them thinking that way.

4.6.3 The Students' Perspective

During their focus-group interview, several students commented on Mr Jones' explanation of $8 - x^2$ as a difference of two squares, when asked what useful things were done by the teacher to help them understand the day's work.

Alan: Oh yeah, the stuff about what happens when there is a difference of two squares, but they were not perfect squares. Yeah, with 8 you can make it so it is root eight squared. I hadn't thought about it before but as soon as he [Mr Jones] said it I'm like "Oh yeah" that's a really logical thing.

Simon: I don't think we've ever done that before when it's not say 9 or something that's easily found.

Danny: Yeah if he [Mr Jones] hadn't done that on the board I wouldn't have thought of it.

4.6.4 Commentary

Mathematical structure and connections was evident when Mr Jones assisted his students to recognise that the "difference of two squares principle" generalises to cases involving non-square numbers during the lesson. His use of the familiar difference-of-two-squares example $9 - x^2 = (3 - x)(3 + x)$ to encourage the students to recognise that $8 - x^2$ can be expressed $(\sqrt{8} - x)(\sqrt{8} + x)$, was coded as *knowledge of examples*.

The post-lesson teacher interview provided evidence of *knowledge of examples* and *anticipation of complexity* when Mr Jones explained that he had selected the item because it was likely to challenge the students' current understanding of the difference of two squares. The tension that Mr Jones expressed in relation to deciding how quickly to intervene when students are presented with challenging mathematical situations, was coded as *anticipation of complexity*. In addition, *theoretical underpinning of the pedagogy* was relevant because Mr Jones articulated tensions associated with his own views on how quickly a teacher should intervene when students are experiencing difficulties with mathematical idea.

4.7 Scenario 6: Using CAS for Integral Calculus

Table 4.6

Information relating to Scenario 6

Date of Lesson Teacher Topic	1st September 2014 Mr Jones Applications of integral calculus to finding areas.
Rationale for selecting the data used to construct this scenario	Illustrates the teacher's functional use of CAS technology to solve a routine example.
Elements of PCK	<u>Concentration on procedures</u> Foundation (KQ) <u>Teacher demonstration</u> Transformation (KQ) <u>Use of instructional materials</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Knowledge of Assessment</u> Pedagogical knowledge in a content context (Chick et al. framework)

This scenario is based on part of a lesson on the application of integral calculus to finding areas bound by functions. The following lesson excerpt illustrates Mr Jones' use of CAS technology to demonstrate the solution to a problem that involved sketching a specific function and calculating the area of some region bound by the curve and the *x*-axis.

4.7.1 Lesson Excerpt

The students had spent the first part of the lesson solving problems from their text-book that involved using integral calculus to find areas bound by functions, without the use of technology. In the second half of the lesson Mr Jones turned their attention to the use of the CAS calculator to find areas using definite integrals.

Mr Jones: Let's have a look at this next one which we are going to do with the calculator [Mr Jones refers to the item in Figure 4.15].

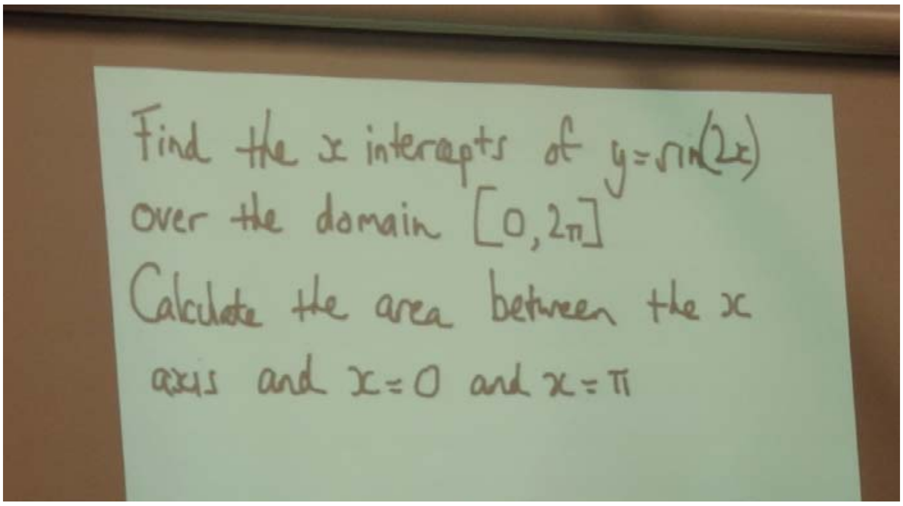


Figure 4.15. The problem that students were asked to solve using their CAS calculator

Mr Jones: What I've done is I've drawn my graph and set my axes from 0 to 2π [See Figure 4.16]. See this 6.28 here? That is 2π, and 3.14, that's π. So, I've drawn the graph of $\sin 2x$. What we might do is mark in π and 2π. So, we know that's $\frac{\pi}{2}$ and that's $\frac{3\pi}{2}$ [These points are marked on the graph as shown in Figure 4.16].

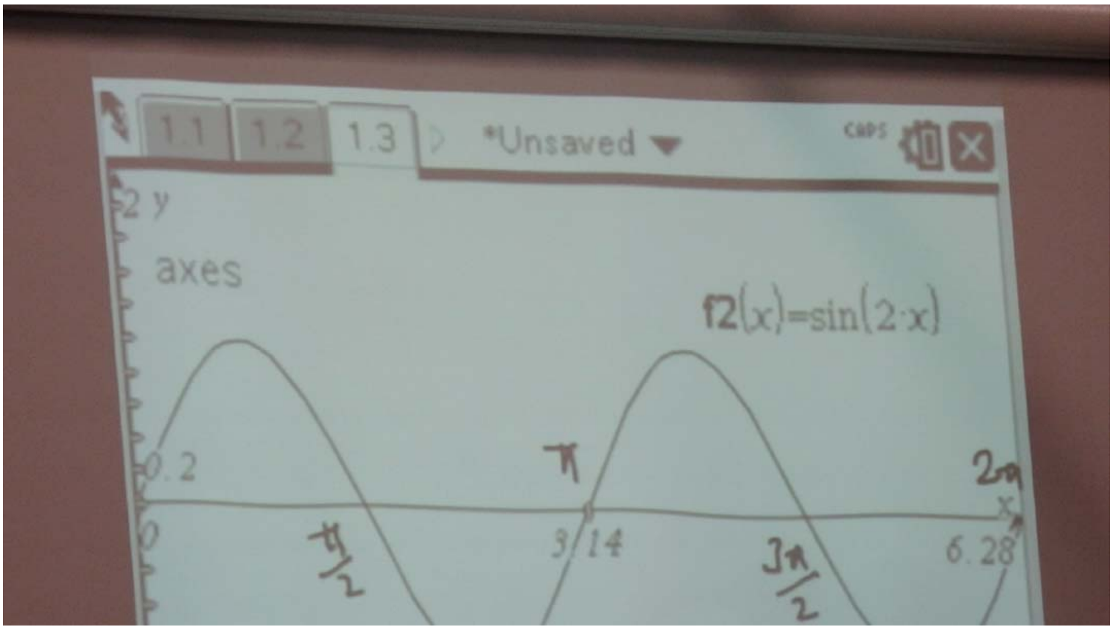


Figure 4.16. Graph of $y = sin\ 2x$ produced by Mr Jones using CAS

Mr Jones: So, the two areas we want here are this area here, and this area here [Points to the orange shaded regions shown on the graphs in Figure 4.17].

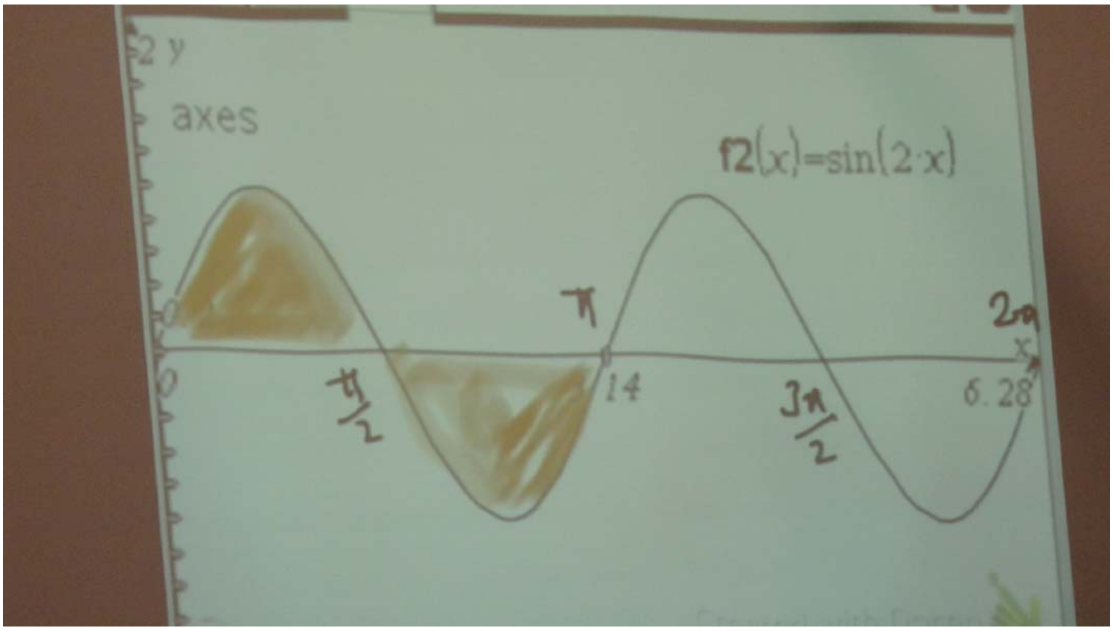


Figure 4.17. Mr Jones indicates the regions in question by shading

Mr Jones then reiterated the problem statement, “We need to calculate the area between the curve and x axis from $x = 0$ and $x = \pi$” and set up the two relevant definite integrals.

Mr Jones: Here are our two integrals [Refers to Figure 4.18].

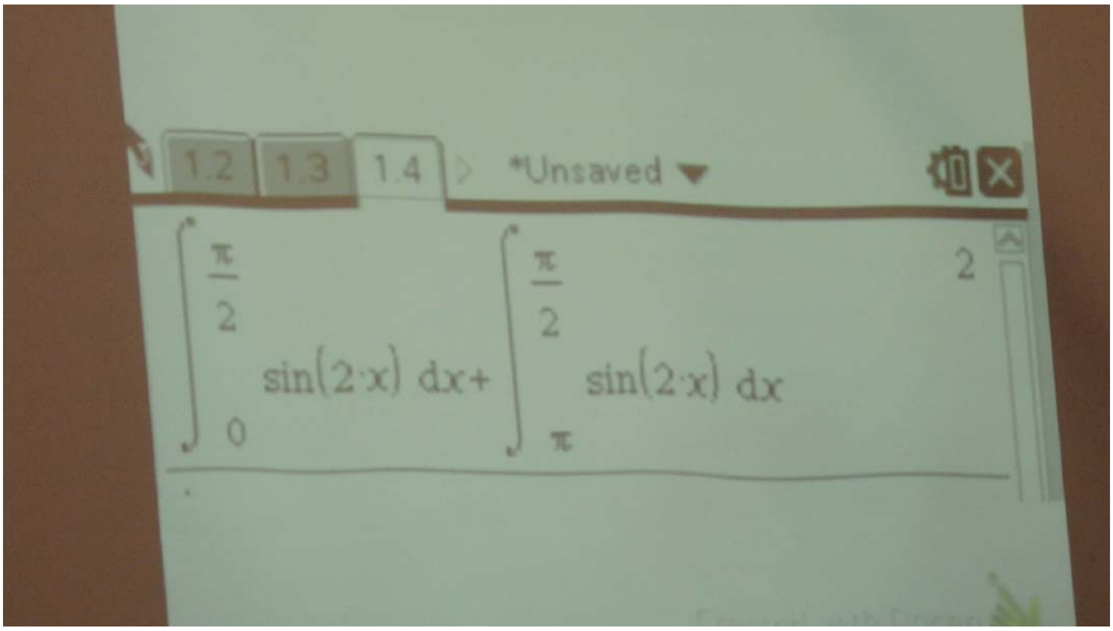


Figure 4.18. The definite integrals representing the areas indicated in Figure 4.17

Mr Jones: For the first one zero is our left-hand terminal, and $\frac{\pi}{2}$ is our right hand terminal and for our [region] below the axis we’ve got $\frac{\pi}{2}$ and π.

The teacher then prompted the students to remember that the area of the region below the x-axis must have a positive value even though the definite integral itself is negative. In the previous lesson the class had been introduced to this idea and shown different ways of ensuring that the area of a region below the x-axis has a positive value by either switching the terminals of the definite integral or using the absolute value of the definite integral.

Mr Jones continued by explaining “Now I prefer to switch the terminals. So, the left terminal is on the top there and the right terminal is on the bottom.” Mr Jones made this explicit by highlighting each terminal on the screen with a green box (Figure 4.19). He then evaluated the definite integrals as follows.

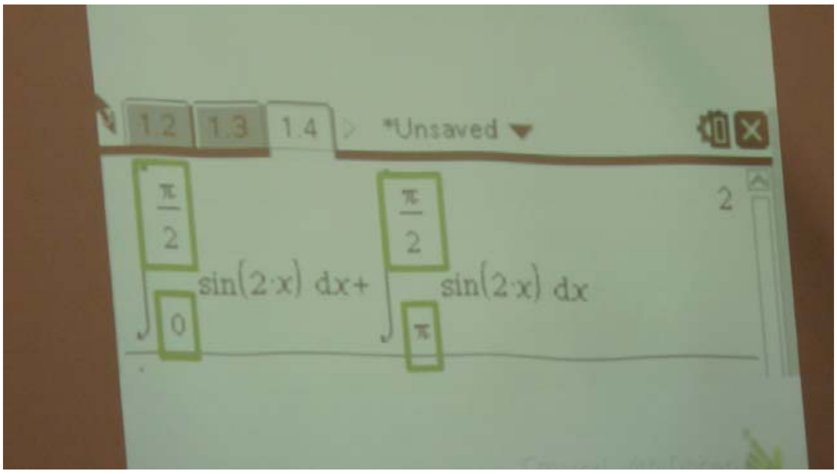


Figure 4.19. Mr Jones highlights the relevant positions of the terminals in green

Mr Jones: In the calculator I’ve set it up as the one question, the integral plus the other integral. We’ve got to add the areas together and the total area is 2 [the answer appears in the top right-hand corner of the screen pictured in Figure 4.19].

4.7.2 Mr Jones’ Perspective

During his post-lesson interview, Mr Jones discussed his focus on the use of CAS for solving some definite integral examples:

Mr Jones: The focus today was very much on the calculator. Because I know that to work out more complex areas, I mean, it is exam driven a bit, but for the more complex ones they are going to be asked to sketch it on the calculator, work out where the areas are, and then get the correct definite integrals to get the area so there is still a lot of maths in that.

He also commented on some of the mistakes that he had observed in relation to the ways in which some students approached the calculation of areas bound by functions, including with respect to the use of the CAS calculator.

Mr Jones: […] towards the end [of the lesson] I had to hold a couple up because there were areas above and areas below [the *x*-axis] and they were still just finding the areas between the two extremes. But no, you can't do that, you've got to work out the area above plus the area below. You can't just take it as a whole thing – so that's just a little thing I've got to watch out for. Then there's the calculator skill and encouraging them to, and I didn't do this on the board, but I went around a few of them and encouraged them to define their function in the calculator to save typing out this complicated thing 3 or 4 times because sometimes they have to do 3 integrals together – there's an above, a below and another above and to type the function in on their calculator it's a time saver. Also, the more buttons they have to press the more the chance is for an error – so just encourage them to define the function once and then just put F(x) in instead of typing out the whole function.

4.7.3 The Student's Perspective

The student interview responses and short written reflections focused on the value of Mr Jones' worked solutions to problems that involved finding areas using integral calculus. For example, James commented:

James: The most helpful explanation was how to find the area under graphs. It helped me to learn how to use integration to solve areas of graphs. (short reflection)

During the focus-group interview, Danny provided the following comment:

Danny: Um well the first couple he [Mr Jones] did were the ones that we were doing [from the textbook exercise] so he went over the different types that you could do and also showing us like shading in what [area] you are trying to find also helps us.

Other students explicitly mentioned the use of the CAS calculator by offering responses such as the following:

Dylan: The most useful thing was the demonstration on how to use the calculator to find areas of integrals. (short-reflection)

Elizabeth: The explanation of how to find areas under the graph – it reminded me how to use the calculator function. (short reflection)

4.7.4 Commentary

Mr Jones' worked solution to the problem in Figure 4.15 was broadly classified as both *teacher demonstration* and *use of instructional materials*. The demonstration was characterised by clear step-by-step instructions on how to calculate the area of the shaded regions using CAS to obtain and evaluate the relevant definite integrals. The teacher used the CAS calculator as a functional tool to model the way in which the students were expected to solve technology-enabled examples involving the application of integral calculus to finding areas. He did not, however, discuss why the area in question is exactly 2 square units by, for example, linking this idea to the fact that the area under the curve of $y = \sin x$ between 0 and π is also exactly 2 square units. As such, Mr Jones' demonstration was also coded as *concentration on procedures*.

The post-lesson teacher interview provided further evidence of *use of instructional materials* when Mr Jones discussed his use of the CAS calculator to demonstrate the application of integral calculus to calculating areas. His focus on the errors that students make, particularly with entering information into the calculator correctly, was coded as *knowledge of students' errors*. Mr Jones' justification for using CAS to solve the problem was classified as *knowledge of assessment* because he emphasised that the students would need to use the calculator in the technology-enabled section of the examination.

The student responses relating to this scenario involved brief statements about the usefulness of Mr Jones' explanations in relation to calculating areas bound by curves and were coded as *teacher demonstration*. Dylan and Elizabeth's responses focused specifically on the use of the CAS calculator to find areas and were coded as both *teacher demonstration* and *use of instructional materials*.

4.8 Scenario 7: "Area under the curve from 1 to *k*"

Table 4.7

Information about Scenario 7

Date of Lesson	16th September 2015
Teacher	Mr McLaren
Topic	Applications of integral calculus to finding signed areas enclosed by functions.
Rationale for selecting the data used to construct this scenario	Highlights a tension raised by Mr McLaren about showing students "interesting things" that happen in mathematics if there is enough time in between covering the course content.
Elements of PCK	Theoretical underpinning of the pedagogy Foundation (KQ) Teacher demonstration Transformation (KQ) Mathematical structure and connections Content knowledge in a pedagogy context (Chick et al. framework)

This scenario is based on a lesson that focused on using integral calculus to find the area of regions enclosed by functions, including those below the *x*-axis. During the lesson, the class were introduced to ways of ensuring that the value obtained for the area was always positive (e.g., switching the terminals of the relevant definite integral). After demonstrating several worked solutions to a range of examples including those involving trigonometric and exponential functions, Mr McLaren sketched the graph of $y = \frac{1}{x}$ on the whiteboard as shown in Figure 4.20. The following teaching and learning episode focuses on the area of the region between 1 and *k* of this function.

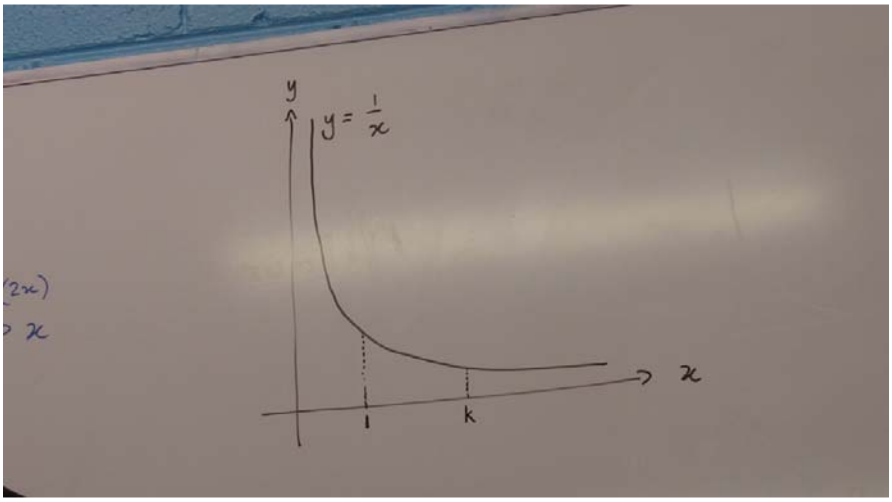


Figure 4.20. Mr McLaren's graph, the feature of this scenario

4.8.1 Lesson Excerpt

Mr McLaren called the students' attention to the graph of $y = \frac{1}{x}$ (Figure 4.20). He shaded the region between 1 and k (Figure 4.21) and told the students that its area is exactly 1 square unit and that he wanted to "investigate something".

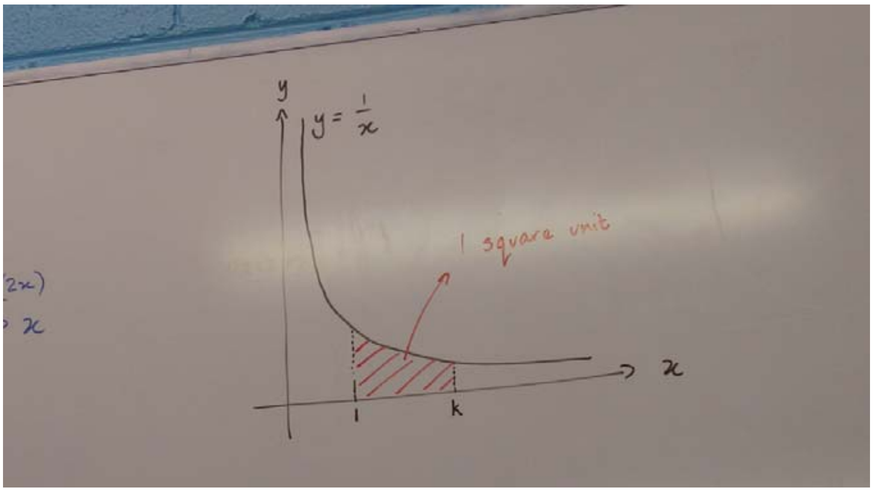


Figure 4.21. Mr McLaren indicates the region between 1 and k

Mr McLaren: Now I want to find out what k is. Don't tell me what it is if you know, but this is quite interesting. So, we are finding the integral between k and 1 of the graph of $y = \frac{1}{x}$ [He set up the relevant definite integral]. And can someone tell me the integral of $\frac{1}{x}$? [Carl responded with "log [base] *e* of the absolute value of *x*"]

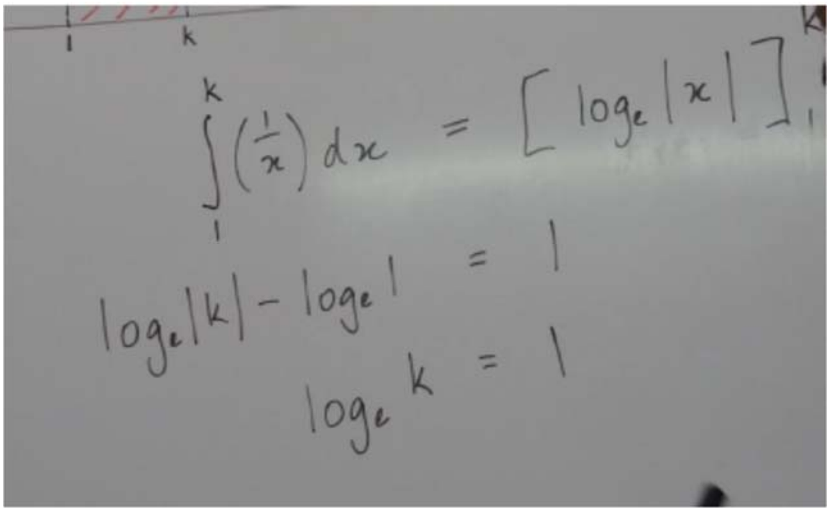


Figure 4.22. The definite integral is equated to 1

Mr McLaren reiterated that the definite integral was equal to 1, performed the substitution of the terminals, and equated this to 1 as shown in Figure 4.22. The students readily acknowledged that $log_e 1$ is equal to zero. Mr McLaren then emphasised that the value of *k* is necessarily positive, and the equation was solved as shown in Figure 4.23.

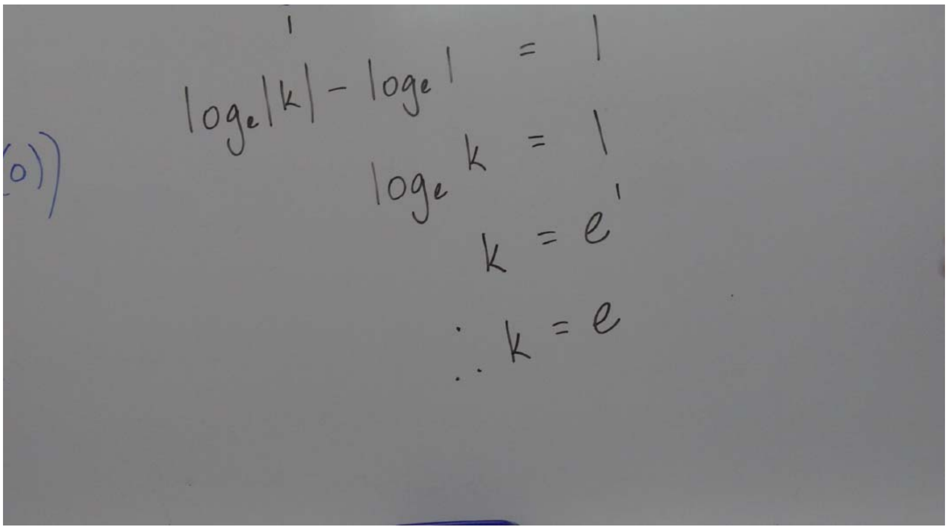


Figure 4.23. The equation involving the relevant definite integral is solved

Mr McLaren: So, have a look at that. The area between 1 and *e* under the curve of $y = \frac{1}{x}$ gives you a value of 1. That's quite interesting isn't it – don't you think? [Some students nodded]

Toby queried why $log_e\ |k| - log_e 1$ is equal to 1 so Mr McLaren reminded him that that were given the information that the area between 1 and *k* was 1 square unit.

4.8.2 Mr McLaren's Perspective

During his post-lesson interview, Mr McLaren elaborated on his decision to show the students that the area under the curve of $y = \frac{1}{x}$ between 1 and e is exactly 1 square unit.

Mr McLaren: Well it's just curious that it works – well I think it is anyway. It's nice for them – to see interesting things that happen even though there's not usually a lot of time to delve into them in any kind of depth. I mean that's quite interesting [refers to the area under the graph of $y = \frac{1}{x}$ between 1 and *e*]. I mean how does that happen? Why does it happen? I like pointing these things out to them because they are interesting.

Researcher: You mentioned there is not a lot of time to delve into things – I wonder if it is a source of frustration that you would like to spend more time on those sorts of things?

Mr McLaren: Oh, I suppose it is but it's just the nature you've got to get through enough stuff to prepare them for next year, for university; it's just the nature of it. So, you deal with it as you can, but you try and build in these little interesting snippets along the way which just happen to be there. I think it's nice for them – to see interesting things that happen. Yeah, I think it's good to make these interesting connections for them.

Mr McLaren continued by reflecting on his approach to addressing the conceptual link between area and integration.

Mr McLaren: They can see a little bit of the wonder of maths if you know what I mean? And it sort of happens more when you get into differentiation and integration.

4.8.3 The Students' Perspective

The students' focus-group responses and short reflections focused mainly on aspects of the lesson that directly involved learning how to solve the standard items from the textbook. For example, several students highlighted the usefulness of learning how to ensure that the value obtained for the area was always positive, and others focused on worked solutions to specific examples. It is not surprising – and a similar situation is discussed in Scenario 12 – that the students tended to prioritise skills acquisition given that they are learning new and challenging mathematics content.

4.8.4 Commentary

A combination of *teacher demonstration* and *mathematical structure and connections* were used to classify Mr McLaren's exposition of the area between 1 and *k* of the graph of $y = \frac{1}{x}$. *Teacher demonstration* was evident in his detailed step-wise explanation of the processes involved in setting up, equating the relevant definite integral to 1, and solving for *k*. *Mathematical structure and connections* was relevant because Mr McLaren sought to direct the students' attention to the "interesting" feature that the area of the region is exactly 1 square unit.

In his post-lesson interview Mr McLaren discussed his decision to investigate the area between 1 and *e* of $y = \frac{1}{x}$. His focus on the idea of sharing "interesting things" that happen in mathematics and his expressed belief that it is "good" for students to see these kinds of intriguing connections, were identified as *theoretical underpinning of the pedagogy*. Interestingly, Mr McLaren's responses also alluded to a perceived dichotomy between taking the opportunity to address the "interesting things that happen" in mathematics and covering "enough stuff" to prepare them for future studies of mathematics.

4.9 Scenario 8: “Trig pops up everywhere”

Table 4.8

Information relating to Scenario 8

Date of Lesson	29th August 2014
Teacher	Mr Jones
Topic	Integral calculus: Evaluating definite integrals in a range of contexts
Rationale for selecting the data used to construct this scenario	Teacher highlights connections between topics of the course. The teacher discusses perceived tensions around prioritising the application of mathematical skills and concepts with limited focus on their conceptual underpinnings.
Elements of PCK	<u>Knowledge of assessment</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Knowledge of curriculum</u> Clearly PCK (Chick et al. framework) <u>Teacher demonstration</u> Transformation (KQ) <u>Knowledge of examples</u> Transformation (KQ) Clearly PCK (Chick et al. framework) <u>Anticipation of complexity</u> Connections (KQ) Clearly PCK (Chick et al. framework) <u>Classroom techniques</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Mathematical structure and connections</u> Content knowledge and a pedagogical context (Chick et al. framework)

This scenario explores aspects of a lesson that involved evaluating definite integrals, including solving equations that involved the use of definite integrals, including those relating to a range of different functions including trigonometric and exponential.

4.9.1 The Lesson Excerpt

Mr Jones began the lesson by demonstrating the solution to the item shown in Figure 4.24. He introduced the item as "a question of the type that crops up each year in the exam so it's pretty important":

If $\int_k^{\pi} \cos(2x)\, dx = -\frac{\sqrt{3}}{4}$, find the value of k given that $0 < k < \frac{\pi}{2}$

Figure 4.24. Problem from Hodgson (2013, p. 427)

As he copied the example onto the board he commented that "this one involves things we haven't done for ages", referring to trigonometric functions, a topic the students had studied earlier in the year. He guided the class through the example as follows:

Mr Jones: Alright now we shouldn't get daunted by this [Points to the item shown in Figure 4.24], we should just do what we usually do and find the integral and see what forms out of it. We'll set up our square brackets with our terminals *k* and pi and we'll worry about the negative root 3 on 4 later. So, we have to get the integral of $\cos 2x$. Now is the integral of cos positive or negative sine? [He asks Lucy and she correctly responded with positive sine. The definite integral was then evaluated]. So, we are told that the definite integral equals negative root three on four. So, can we find the value for this [Points to sin2π]. What's the value of sin2π, which is sin360? [Someone says: "Umm is it zero or one?"]. Ah is it zero or one? That's important, if you're unsure you do have this guy here right in front of you [He holds up the formula sheet, an official course document which includes a table of basic angles – the students are able to take this formula sheet into the external exam]. I encourage you to go from memory because it does save a bit of time – but if there's the slightest doubt please refer to this, it's there for that reason [Mr Jones leads the class through the simplification of the equation as shown in Figure 4.25].

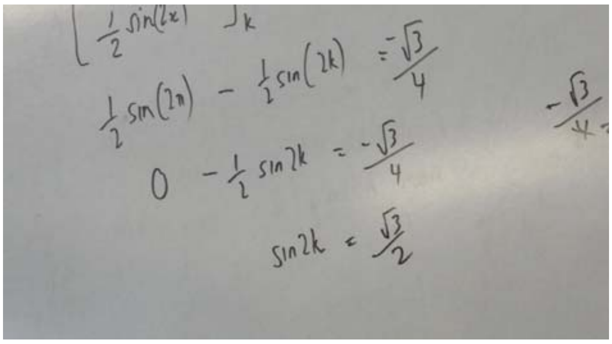


Figure 4.25. Screenshot showing part of Mr Jones' solution to the trigonometric equation in question

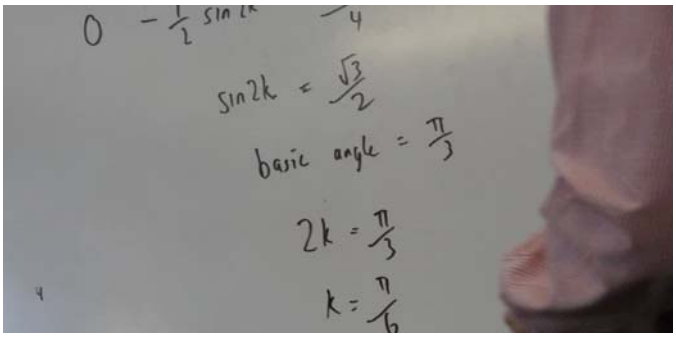


Figure 4.26. Screenshot showing resulting value of k from the equation in question

Mr Jones: Now remember back to trig equations – and this is a pretty simple one because we're only after the acute angle, zero to 90 – James what do we need to find at this point? One word starting with b and the other with a.

James: Basic angle.

Mr Jones: Basic angle – good. Ok and what is that basic angle? It is the angle that has a sin value of $\frac{\sqrt{3}}{2}$. Now once again we've got our table [on the formula sheet] in front of us. What angle has a sine value of $\frac{\sqrt{3}}{2}$? [Someone responds with $\frac{\pi}{3}$ and the equation was solved for *k* as shown in Figure 4.26]. So, you know in an exam that would be worth quite a few marks that question, but it's a matter of taking one step at a time. I don't know how many times in exam situations where you're asked to find *k* somewhere across lots of topics and you can get concerned about it but just go back to the basics that you know, you know how to integrate. OK, so the *k* is still there, that's fine, it stays there, right the way through. So just break it down into skills that you know how to do, and you will get there in the end.

4.9.2 Mr Jones' Perspective

Mr Jones elaborated, in his post-lesson interview, on his rationale for focusing on an example which involved a trigonometric function:

Mr Jones: My considerations today were, and I named it up because I wanted to choose some examples that concentrated a bit more on trig [trigonometry] and brought in some other skills from earlier in the year because the thing with the Methods course is that trig pops up everywhere in lots of topics. The only topic it doesn't pop up in is probability. But it pops up everywhere else and my thought today was to choose examples of using *e* and also trig rather than just concentrate on *x* expressions.

In response to a question from the researcher about whether Mr Jones had "done anything different this year with his approach to definite integrals", the teacher elaborated on recent changes to the way in which he had introduced the topic of integral calculus. Although the following responses are not directly related to the lesson excerpt featured in this scenario, they provide interesting insight into other aspects of Mr Jones' PCK.

Mr Jones: This year in integral calculus I did more of the rectangle method [a numerical method of approximating the area enclosed by a function]. I did it once about three or four years ago; I mean it's not technically in the course.

Researcher: What did you see is as the key purpose for doing this?

Mr Jones: I think it's, I mean it is related to the fundamental theorem [of calculus] which I don't do very deeply. I know there is a whole proof about how you … well, you know. It's probably naughty of me but it's quite, it is not an easy thing to understand and I don't think, for the purpose of our course and finding areas, I don't think spending quite a large amount of time on it [the Fundamental Theorem of Calculus] is needed. And I doubt even if I did that whether there would be full understanding about how the theorem works, or whether it's worth the time and the effort. Why did I do the rectangles? Well, because I wanted to show them that there is an alternative way of finding areas, but also that it is very inefficient and there are more accurate ways. Ideally, I'd love to make that link between the rectangle method and finding definite integrals, but I didn't.

I did try it once four years ago and I don't think the kids ultimately got very much out of it. I probably, deep down, I probably feel like I should but the experience of when I did it last time working through the $f(x + h)$ and everything well mm. The course really is focused on finding areas between curves, under curves and in real life situations as we'll get on to in a week or two. So, I could spend a period doing it [the Fundamental Theorem of Calculus] and I could make the link between the rectangle method and that, but I chose not to because I don't think it's worth the 45 or 50 minutes and whether they'd all perhaps cotton on to it either. There are probably lots of topics where I could go into the background of things a little bit more than what I do, but I've got a 150 hour course that I'm trying to teach in 110 [hours] so you know there are some things I've just got to let go. Ultimately, I don't think their ability to find areas and all the other things they're going to have to do in the next couple of weeks will be any the poorer for not having done it [the Fundamental Theorem of Calculus].

4.9.3 The Students' Perspective

During the focus-group interview one of the students remarked on the combination of trigonometry and integral calculus in the one item:

Dylan: I thought that when you combine the trig equations to the integral bit that was like – what!? I'd never done any of that before, so I learnt about that.

Another student, Amy, made the following comment in her short reflection:

Amy: The most helpful examples were the white board examples that recapped on old methods – like we used basic angles with trig.

Similarly, Elizabeth recorded that:

Elizabeth: Doing more questions on the board helps [me] realise we need to remember all of our past chapters even when doing integration. It helped me learn how much we need to remember. (Short reflection)

4.9.4 Commentary

Mr Jones' worked solution to the problem in Figure 4.24 was broadly coded as *teacher demonstration*. Evidence of *knowledge of examples*, *knowledge of assessment*, *anticipation of complexity*, *classroom technique*, and *mathematical structure and connections* were evident within the demonstration itself. *Knowledge of examples* and *knowledge of assessment* was demonstrated when Mr Jones explained to the class that he had chosen the example based on the likelihood that a similar one may appear in the final examination. His identification of the features of the example that were likely to challenge the students (i.e., the combination of the use of basic angles of trigonometric functions and integral calculus in the one equation) was coded as *anticipation of complexity*. The use of funnelling questions to prompt students to recall relevant information (e.g., "What angle has a sine value of $\frac{\sqrt{3}}{2}$?") was classified as *classroom techniques*.

The post-lesson interview provided further evidence of *knowledge of examples* when Mr Jones explained that he had chosen a range of examples including the one featured in this Scenario, so that he did "not just concentrate on *x* expressions". He also highlighted the fact that "trig pops up everywhere" in the course and it is therefore important to provide students with experiences that combine skills from a range of topic areas. This aspect of Mr Jones' interview was coded as *mathematical structure and connections*. His justification for choosing not to focus on the connection between the "rectangle method" and the Fundamental Theorem of Calculus course was classified as *anticipation of complexity* and *knowledge of curriculum*. To some extent this *anticipation of complexity* also seemed to be linked to Mr Jones' own lack of confidence in his capacity to teach the Fundamental Theorem of Calculus, based on his perceptions of a previous teaching experience. *Knowledge of curriculum* was evident when Mr Jones suggested that the integral calculus component of the course focused primarily on the application of skills to practical contexts. He also emphasised the time constraints associated with covering the required course content.

The focus-group interview and short reflections suggest that several students appreciated Mr Jones' worked solution to the problem in Figure 4.24. Lucy's response about the value of the teacher's worked examples in "recapping on old methods" was coded as *teacher demonstration*. Other comments, including those from Dylan and Elizabeth, suggested that Mr Jones' demonstration had "brought home" for them the need to incorporate content from previous mathematics topics with the current topic and were therefore coded as *mathematical structure and connections.*

4.10 Scenario 9: “e^2 is just a number”

Table 4.9

Information relating to Scenario 9

Date of Lesson	14th August 2015
Teacher	Mr McLaren
Topic	Integral calculus: Anti-differentiating exponential functions
Rationale for selecting the data used to construct this scenario	Students discuss the way in which their teacher empowers them to identify and overcome difficulties when solving mathematics problems.
Elements of PCK	Knowledge of students’ errors Foundation (KQ) Theoretical underpinnings of the pedagogy Foundation (KQ) Teacher demonstration Transformation (KQ) Knowledge of examples Transformation (KQ) Clearly PCK (Chick et al. framework) Classroom techniques Pedagogical knowledge in a content context (Chick et al. framework)

This scenario focuses on part of a lesson that focused on anti-differentiating trigonometric and exponential functions. Mr McLaren began the lesson by introducing the integral $\int e^x dx$. He asked the class to recall that the derivative, $\frac{d}{dx}e^x$, is equal to e^x hence the integral $\int e^x dx$ is equal to e^x+ c, where c is the constant of integration. This was followed by the derivation of the rule $\int e^{kx} dx = \frac{1}{k}e^{kx}$+ c since

$\frac{d}{dx} e^{kx} = ke^{kx}$. Mr McLaren then guided the students through a sequence of worked examples involving integrating exponential expressions.

4.10.1 The Lesson Excerpt

The lesson excerpt focuses on the example $\int(2e^{-2x} + e^2)\, dx$.

Mr McLaren: Talk me through this one here [Points to the integral $\int 2e^{-2x}$ + $e^2)$ dx on the white board]

Mr McLaren: You said something before Grace?

Grace: Yes, so just put it into two so the integral of the first half plus the integral of the next bit.

Mr McLaren: Ok, so that's a first step. Now I would be happy if you did not put this line in [points to $\int 2e^{-2x}$dx + $\int e^2\, dx$] but if it makes you feel comfortable about doing your integrals then fine but I wouldn't mind if you went straight to the next step. Ok so what now?

Carl: You take the two out. [Mr McLaren encourages him to be more specific] Just pop it out the front.

Mr McLaren: You mean this two [Points to the two in front of the e^{-2x}, and records the following reasoning on the white board]:

$\int(2e^{-2x} + e^2)\, dx = \int 2e^{-2x}dx + \int e^2\, dx$

$= 2\int e^{-2x}dx + \int e^2\, dx$

Carl: [Refers to the first integral in the sum above, $2\int e^{-2x}dx$] So now you can make that 2 times minus ½ *e* to the 2*x*. [He then pauses as he seems to contemplate the next integral, $\int e^2\, dx$. Others then begin to join in. The following part of the transcript involves the students grappling with $\int e^2\, dx$].

David: It's ½ um

Kale: Surely *e* to the *x* equals *e* to the *x?*

Toby: But there's no *x* up there.

Kale: Oh yeah there isn't.

Mr McLaren: There is no *x* is there. So, what is *e* squared?

David: Is it a third *e* cubed. Is that right?

Mr McLaren: No [There is some mumbling as students seemed to ponder this].

Grace: Wouldn't it be *e* squared *x*? [Mr McLaren nods and writes *e* squared *x* [e^2x]and there were audible exclaims of "Oh yeah, I see!"]

Mr McLaren: Why? [Again, there is some mumbling and then Grace says: "Because that's actually a number."]

Mr McLaren: That's right, *e* squared is just a number, it is not a variable it's just a number. OK? Approximately what number? [There is some mumbling and offering of numbers] It's approximately 7 point something or other.

4.10.2 Mr McLaren's Perspective

Mr McLaren elaborated on his approach to demonstrating the example featured in the lesson excerpt during post-lesson interview:

Mr McLaren: It has probably only come through experience but knowing the pitfalls that they'll come across or that they'll fall into. And one of the examples that I gave, I knew that this was a common mistake. I've probably dealt with it informally in the past, like later on or a few lessons later or after they've finished the assignment. But this time I thought "I'll put this in there", that was the e^2one. So that's only come through the experience of knowing what mistakes they'll make. I wonder whether knowing those kinds of things and knowing they are going to make those kinds of mistakes and, you know how you learn out of confusion, so if you can be pointed or if you can be taken through it from confusion to understanding, then it is going to be more valuable. It's going to be a valuable experience if they can see that they would have made that mistake and then be shown what the correct way to go is. I think that's useful. And I think I wanted to let them make the mistake here and then talk about it. I guess that has only come with experience. Well I've seen that same mistake being made reasonably regularly.

4.10.3 The Students' Perspective

During their post-lesson interview two students offered their insights into the way in which Mr McLaren guided their thinking during the lesson:

Kale: I think a good tool that he [Mr McLaren] uses and he probably doesn't even realise he uses it – is like sometimes when he's teaching he asks someone to go through it and then someone sort of like takes the lead and they try to have a go at it then he sort of like watches them and then almost like lets them fail. Like the way that he almost lets you fail is like a little bit of embarrassment, like you sort of feel a little bit sort of "Oh I should have known that" and so then you go you think to yourself and then the embarrassment sort of embeds it in your mind a little bit more. Sort of the fact of being embarrassed sort of pushes it from short term memory to long term memory.

Researcher: So, it's not like taking the hurdle away from you?

David: It's like how you handle the hurdle. In the working and how you think through it. Just with him letting you get a bit stumped on your answer – it's a good way of enforcing the principles.

Kale: Yeah, getting you to nut it out a bit before he prompts you is probably a good thing because he can be doing an example on the board and we'd be going cross eyed having no idea what is going on and then he'll like say a couple of things like "what do you do with this?" and then everyone's like "Oh yeah [Taps his forehead] duh no wonder. And from there, like that one little prompt, sometimes we can just go through and do the entire question by ourselves.

David: Sometimes it's just one little snippet of the question

Kale: He must understand what's stumping us, what the small little part of the question that's stumping us

David: Yeah understands our thought process towards the question.

4.10.4 Commentary

Mr McLaren's worked solution to $\int(2e^{-2x} + e^2)\,dx$ was classified as *teacher demonstration*. The way in which he encouraged the students to grapple with the problem to come to the realisation that e^2 is a number and not a variable to be integrated was coded as *classroom technique*.

The post-lesson interview provided evidence of PCK that corroborated the researcher's observations and the students' responses and shed further light on Mr McLaren's instructional decisions. *Knowledge of examples* and *knowledge of students' errors* were evident when the teacher explained that he had selected the item (i.e., $\int(2e^{-2x} + e^2)\,dx$) based on his experience of the errors that students typically make. Mr McLaren's comments about encouraging his students to confront their own errors to enable them to move from "confusion to understanding" were coded as *classroom techniques*. In addition, his suggestion that students learn more effectively when they confront and grapple with their own errors was coded as *theoretical underpinning of the pedagogy.*

The student focus-group interview responses corroborated the researcher's observations and the teacher's justification for his teaching approach. Kale and David made particularly insightful comments relating to the way Mr McLaren allowed them to "get a bit stumped" and how this was a valuable precursor to their knowledge growth. These comments were coded as *classroom techniques* and also reflect that the students, to some extent, recognise *theoretical underpinning of the pedagogy*.

4.11 Scenario 10: Introduction to Variance

Table 4.10

Information relating to Scenario 10

Date of Lesson	16th September 2015
Teacher	Mr McLaren
Topic	Probability: Introduction to variance
Rationale for selecting the data used to construct this scenario	The teacher constructs an example designed to illuminate a specific mathematical idea and the students do not readily grasp the ideas intended. The teacher calls upon on his own mathematical content knowledge in response to an unexpected question about a concept that is not explicitly part of the course.
Elements of PCK	Overt display of subject matter knowledge Foundation dimension (KQ) Concentration on procedures Foundation (KQ) Knowledge of curriculum Clearly PCK (Chick et al. framework) Teacher demonstration Transformation dimension (KQ) Knowledge of examples Transformation (KQ) Clearly PCK (Chick et al. framework) Knowledge of representations Transformation (KQ) Clearly PCK (Chick et al. framework) Knowledge of instructional materials Transformation (KQ) Clearly PCK (Chick et al. framework) Classroom technique Pedagogical knowledge in a content context (Chick et al. framework) Responding to pupil ideas Contingency (KQ)

Deconstructing mathematics into key components
Content knowledge in a pedagogical context (Chick et al. framework)

This scenario is based on a lesson that began with an introduction to the concept of variance, followed by the completion of examples that involved calculating the variance of probability distributions.

4.11.1 The Lesson Excerpt

Mr McLaren began the lesson by showing graphs of two different discrete probability distributions designed to illustrate the concept of variance (see Figure 4.27.). The distributions were symmetrical and had the same mean (expected value), median, mode, and range, but different variances.

Figure 4.27. Graphs of two discrete random probability distributions chosen by Mr McLaren

The students were asked first to identify the similarities between the two distributions:

Kale:	Well it goes from one to ten for both of them [the *x* values for each distribution].
Toby:	They are both symmetrical
Kale:	They both have a median of 5.5.

David: They both peak in the middle. [Mr McLaren acknowledges that "all of the above are correct"]

Mr McLaren: What can you tell me about the mean, the expected value? [Mumbling from class]

Kale: I can't really see the numbers, Oh yes I can, I don't reckon they would be the same because they've got three under 0.05 for the blue and two under for orange.

Mr McLaren: Are you sure?

Kale: Oh, wait I reckon they might be the same because the blue has three under 0.05 and there's two under the orange and the blue ones has a higher peak so I reckon so it should actually average out to be similar.

Some students appeared to have difficulty interpreting the mean of the distributions, so Mr McLaren responded by explaining:

Mr McLaren: It actually doesn't matter what the probabilities actually are, the fact that it's symmetrical will mean that the expected value or the mean is 5.5 in both cases. So, it is the case that the mean or the expected value is the same in both these distributions.

When asked to consider the differences between the two distributions, the students gave valid responses based on what they saw; for example, "there are different probabilities for *x* in each one" (Toby), and "the peaks are higher in the blue, but then it stays lower in the blue. It's kind of more exponential in the blue" (Kale). While these comments alluded to the blue distribution having a smaller spread as it is concentrated around the mean, the students did not appear to make this link readily, so Mr McLaren intervened as follows:

Mr McLaren: All of those things you have said are correct. The thing I wanted you to see and perhaps you have thought about this, is that the probabilities in this one [Points to the orange graph in Figure 4.27] are spread out more than in this one [Spans hands around the blue distribution]. The probabilities in this particular distribution [blue] are really focused around the centre whereas the probabilities in this one [Points to orange distribution] are been spread out more. Can you see that?

Following the comparison of the graphs Mr McLaren highlighted the idea of the spread of the distributions and then introduced the variance. Mr McLaren described the variance as "a measure of each of the values of x and how far they are away from the mean, squared, and multiplied by their associated probabilities" and presented the formula Var(X) = $\sum(X - \mu)^2$ Pr(X =x). He then used this formula to model the process of calculating variance using the data from each of the two original distributions.

Mr McLaren: We will talk about this part of it first [Points to the (x - μ) part of the above-mentioned formula, he then returns to the previous examples involving the orange and blue distributions]. So, we said before that the mean is 5.5. This here [Points to the (x -μ) part of the formula] signifies the difference between each x value and the mean. So here we've got [Points to each x value in turn on the orange distribution as shown in Figure 3.28] 5 minus 5.5, 4 minus 5.5, 3 minus 5.5, 2 minus 5.5 and 1 minus 5.5.

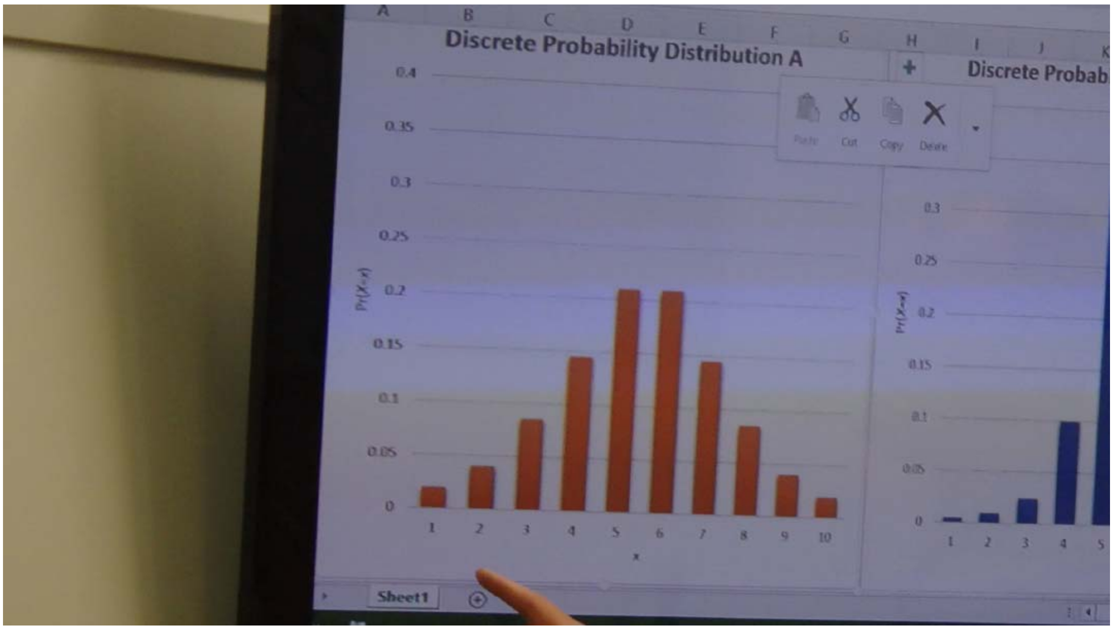


Figure 4.28. Mr McLaren introduces students to the idea of the difference each *x* value is from the mean.

Mr McLaren: And then it goes back the other way. So what values will we get? If I'm just looking at the inside of this [meaning the (x - μ) part of the formula] we will get −0.5, -1½, -2½, -3½, -4½ and we'll also get positive values because we've got 6 minus 5.5, 7 minus 5.5 and so on. Why do you think we'd square it? [Points to the $(x - \mu)^2$part of the formula]

Toby: Because you don't want negatives.

Mr McLaren: Good. Yes – we don't want to have negative values, so we square it – OK. So that will mean that each of these will be 0.5 squared, 1.5 squared and so on, in both directions. So that gets rid of the negative values because if we had the negative values in either of those two examples [Points to the orange and blue distributions] then they'd cancel each other out and we would end up with a variance of zero which would be useless. Then once we've squared it, we multiply it by its corresponding probability, so here it's about 0.22 [Points to the

5th column from the left on the orange distribution], 0.14 [Points to the 4th column from the left on the orange distribution] and so on. So, we multiply them by their associated probabilities and add them together, and that gives us a measure of how the distribution is spread out.

While the formula Var(X) = $\sum(X - \mu)^2$Pr (X=x) explicitly articulates the process of obtaining the variance of a discrete probability distribution, it is quite cumbersome to use. Therefore, an alternative equivalent formula, albeit less intuitive in terms of capturing the idea of values varying from the mean, is commonly used for calculating variance: Var(X) = E(X^2) - [E(X)]^2.

Mr McLaren: There is a proof to show you how to get from here to here [Points to Var(X) = $\sum(x - \mu)^2 Pr(X{=}x)$ and then to $E(X^2)$ - $[E(X)]^2$] but I'm not going to worry about that at the moment [as] you don't have to know it. Really what you need to remember is this here [Points to E$(X)^2$ - $[E(X)]^2$].

He explained the use of the formula by demonstrating the solution to several worked examples on the whiteboard. The students then completed a text-book exercise involving problems such as the one shown in Figure 4.29.

5 A random variable has the following probability distribution.

x	2	4	6	8
Pr($X = x$)	0.15	0.3	0.42	0.13

Find (a) Var(X)
(b) Var($2X$)
(c) Var($3X + 1$)
(d) Var($-5X + 7$)

Figure 4.29. A question from a class exercise from the prescribed text-book Hodgson (2013)

The exercise required the students to calculate the variance of a linear function of a discrete random variable, such as Var($2X$) which involved the use of the relationship Var($aX + b$) = a^2Var (X).

Mr McLaren: If you're asked to do something like this [Writes the rule Var(aX+b) = a^2Var(*X*) on the board] […] you do the following [Writes Var (2X) is equivalent to 2^2Var (*X*)].

Grace: What about the *b* up the top?

Mr McLaren: It's just gone.

Grace: Really! [She sounds amazed]

Mr McLaren: Yeah. [He pauses for a couple of seconds] We could look at why but I'm not too fussed. [As the students continue working, Mr McLaren seems to ponder this. He looks down at the open text-book, turns a couple of pages and appears to be thinking]

Mr McLaren: [Turns to the board again] If you think in terms of what the variance actually is – I said I wasn't going to show you – but anyway [He clears the board leaving the expression Var(*aX*+*b*) and proceeds to express Var(*aX*+*b*) in the form Var(*X*) = E(X^2) – $[E(X)]^2$. He hesitates a little, checks the text-book, and writes the following: Var(*aX*+*b*) = E$(aX + b)^2$ - $[E(aX + b)]^2$ = a^2E(X^2) + b^2 - $[E(aX + b)]^2$]

Mr McLaren did not attempt to deal with the $[\mathrm{E}(aX + b)]^2$ component but explained that "in the end what will happen is we end up with minus *b* squared on the end, so the *b* squared terms end up cancelling out. I don't think you're going to come across this at all".

4.11.2 Mr McLaren's Perspective

At the beginning of his post-lesson interview, Mr McLaren remarked that although "probability seems the shortest unit to teach and the most straight-forward in some respects", it involves concepts that he considered to be difficult to "explain in a way that make sense to students”. He identified the concept of variance, and the way in which variance is calculated, to be particularly challenging to learn and teach. When invited to elaborate Mr McLaren provided the following response:

Mr McLaren: I can see it’s [the variance] a measure of spread, and I can explain to them that it is a measure of spread, but why do you choose the squares of the differences? Why that [the squares of the differences] as opposed to some other way of using the differences? [Mr McLaren is referring to the difference each score is from the mean] I guess, I mean, yeah, you can definitely see why variance looks different on a graph and that’s what I tried to do today – but I still I think it’s not quite as concrete as some of the other concepts that you can teach students.

The researcher recalled that Mr McLaren had asked the students why the differences from the mean are squared:

Researcher: You asked the students why the differences from the mean are squared and Carl responded that it was because you don’t want negative numbers and you elaborated on this further. Yet you also express concern about the reasons behind why the variance deals with these squared values. Are you able to explain this concern a bit further?

Mr McLaren: Well I think it’s just probably because I don’t know why. I mean you can see that the greater the value for the variance, you can conceptually see what effect that would have on the distribution yeah. But I figure I just need to look into it a bit further. I just think that the students might wonder why the squares of the differences have been chosen. You can see because it gives you a positive value and you can see that the difference from the mean doesn't add up to give negative results, but still why a squared value?! Yeah, I don't know I probably need to look into it so that I can actually give an explanation because even though you don't often get asked, understanding why is always very useful in understanding what you're doing I think. I think when it boils down to it the main thing is that I just don't know enough about it myself.

Mr McLaren also elaborated on the considerations he had taken into account when he developed the two data sets to introduce the idea of variance:

Mr McLaren: I wanted to have enough values for *x* but not too many, like *x* from 1 to 30. I didn't want to have that many values but then I didn't want to have too few values because then the spread wouldn't be as obvious, so I just wanted a bit in between. I wanted to have one [graph] with a wide spread and one with a narrow spread around the mean. Both with pretty much everything else the same so they had the same mean, the same *x* values so everything is the same, but you can see that the probabilities were spread narrowly with one and spread more widely with the other. Then you can get an idea that spread might be important.

Later in the interview, Mr McLaren focused again on the challenges involved with teaching probability, from his perspective:

Mr McLaren: […] They [the students] don't have any problem with measures of centre, so mean, mode and median, because they are commonly used, particularly the mean. We talk about the average of this or that all the time, but we don't usually talk about how data is spread. So it's easy enough to talk about say the mode being the most frequently occurring x value with the highest probability, that's pretty easy. And the median being the middle value, that's also pretty easy to grasp. And the idea of having something you can compare yourself against is what they do all the time. They are comparing things against averages all the time. They are comparing their test marks against averages or whatever. But measures of spread is well, I suppose the difference is when you are talking about average they are comparing one piece of data, their own piece of data against the average as opposed to spread where you are talking about a whole population, maybe that's why they don't get it as much.

He also discussed his approach to explaining why the 'b' "disappears" in the $\text{Var}(aX + b) = a^2\text{Var}(X)$ and reflected on how he may have improved the explanation:

Mr McLaren: I'm figuring out a way of explaining it better. I think I was fumbling around a bit there but yeah but because 'a' is having a multiplying effect on all the 'x' values and 'b' is having an additive effect. When it comes to variance you're not worried about the additive effect because the spread is all added by that […] 'b' so the variance doesn't get changed by the 'b'. So 'b' has no effect on the variance which is a measure of spread. The spread remains the same so that's why the 'b' disappears. But the 'a' does have a multiplying effect and because variance is about a square of the differences the 'a' has got to be squared.

4.11.3 The Students' Perspective

During the focus-group interview, one student (Grace) commented that she found Mr McLaren's two graphs (Figure 4.27) useful:

Grace: Well I liked how he did the graphs because it was sort of a more like a visual representation like you saw that – Oh – it's not just numbers.

Researcher: How did that help you?

Grace: You could sort of see the shape of the data, what it was doing. It sort of like compared more spread out numbers to like all close together ones, or concentrated ones. It sort of gave a bit more motivation [Giggles]. It was like "Oh we have a shape; it's not just numbers".

Kale particularly noticed and appreciated Mr McLaren's decision not to delve into why $\mathrm{E}(X)^2 - [\mathrm{E}(X)]^2$ is equivalent to $\sum(x-\mu)^2\mathrm{Pr}(X{=}x)$:

Kale: I think the fact that he [Mr McLaren] decided to skip over the proof of the $\mathrm{E}(X)^2 - [E(X)]^2$ was helpful because it didn't add too much information. He didn't overload us with too much stuff, yeah.

Kale also expressed a similar view in his written response: "The most helpful thing I believe was him skipping over the proof of $\mathrm{E}(X)^2 - [\mathrm{E}(X)]^2$ so as not to confuse us further." In addition, other students commented that learning the processes of calculating variance was the most useful aspect of the lesson. For example, Carl made the following comment in relation to the example involving Var(2*X*) = 2^2 Var(*X*): "The most helpful thing that I learnt was those cases when the variance is like (2*X*), it's not like take the 2 and times it by the *X*, it's like take the 2 and square it and just those small things." Toby recorded the following comment in his short reflection: "The most helpful thing was going through the questions step by step and to follow the exact steps so that you don't make a mistake".

4.11.4 Commentary

Mr McLaren's use of the two probability distributions (Figure 4.28) to develop the idea of variance was coded as *teacher demonstration*, *knowledge of examples*, *knowledge of representations*, and *knowledge of instructional materials*. *Knowledge of representations* and *knowledge of examples* were demonstrated because Mr McLaren was able to construct two data sets with markedly different spreads and represent this data in the form of graphs for comparison. The use of spreadsheet software to generate two data sets showing different spreads was coded as *knowledge of instructional materials.*

The teacher's use of questioning to encourage his students to connect the visible differences between the distributions with the idea of spread was classified as *classroom techniques*. Mr McLaren's approach to explaining the formula Var(X) = $\sum(X - \mu)^2$Pr(X=x) was coded as *deconstructing mathematics into key components* because he unpacked the formula and made explicit connections with the accompanying graphs. In addition, he focused the students' attention on why the differences from the mean are squared, an idea that Mr McLaren again raised during his post-lesson interview.

Mr McLaren's explanation of the alternative formula for variance, Var(X) = E(X^2) – $[\mathrm{E}(X)]^2$ was coded as *concentration on procedures* and *knowledge of curriculum* because he suggested that it was not necessary to address the derivation, as this was not a requirement of the course. Mr McLaren did, however, address the tension between using a formula and explaining why it works. This was evident when he discussed the rule for calculating the variance of a linear transformation of the variable, such as Var(2X). He called upon his own content knowledge in the moment of teaching to address Grace's query and attempted to deconstruct the formula to show why the b "disappears" from the right hand side of the formula $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$. This aspect of the lesson was coded as *deconstructing mathematics into key components*, *responding to pupil ideas*, and *overt display of subject matter knowledge*.

Mr McLaren's interview responses support the researcher's observations of his *knowledge of examples* and *knowledge of representations* in relation to the construction of two distributions that were useful illuminating the idea of variance. He also articulated the challenges of making the concept of variance "make sense to the students". Such challenges were a key theme of the interview because Mr McLaren discussed the complexities of teaching and learning variance both in terms of appreciating it intuitively as the spread of the data, as well as from the perspective of how variance is calculated. His suggestion that measures of centre were more easily grasped by students than measures of spread was coded as *anticipation of complexity*.

Mr McLaren's expressed frustration about how best to explain why differences from the mean are squared in the calculation of variance was identified as *deconstructing mathematics into key components.* In addition, his reflection on his initial response to Grace's query about why the b "disappears" conveys how he was later able to articulate rich connections between the concept of variance and the algebraic processes involved in calculating it. In this sense, Mr McLaren was able to deconstruct the mathematics during his reflection-on-action in ways that were not evident in the moment of teaching.

The student focus-group interview and short reflections offered limited insight into the extent to which students perceived the pair of graphs (Figure 4.27) as helpful for their learning of the concept of variance. Although Grace commented that the graphs were useful as visual representations of the distributions (*knowledge of representations*), the extent to which she associated their differences with the spread of the data is unclear. Several students focused on specific details involved in calculating variance and these were coded as *concentration on procedures.* Kale's comments relating to his appreciation for Mr McLaren's decision not to address "the proof" that $\mathrm{E}(X)^2$ - $[\mathrm{E}(X)]^2$ is equivalent to $\sum(x-\mu)^2\mathrm{Pr}(X=x)$ were also identified as *concentration on procedures.*

4.12 Scenario 11: "It's Pascal's Triangle": Mr McLaren's Introduction to the Binominal Probability Distribution

Table 4.11

Information relating to Scenario 11

Date of Lesson	17th September 2015
Teacher	Mr McLaren
Topic	Probability distributions: Introduction to the binomial probability distribution
Rationale for selecting the data used to construct this scenario	Teacher adapts a textbook activity in ways that he believed would be pedagogically powerful.
Elements of PCK	Theoretical underpinnings of the pedagogy Foundation (KQ) Use of instructional resources Transformation (KQ) Clearly PCK (Chick et al. framework) Mathematical structure and connections Content knowledge in a pedagogy context (Chick et al. framework) Classroom techniques Pedagogical knowledge in a content context (Chick et al. framework)

This scenario focuses on part of a lesson that involved an introduction to the binomial distribution. Mr McLaren began the lesson by introducing the idea of a Bernoulli trial, a random experiment (e.g., tossing a coin or rolling a die) with exactly two outcomes ("success" or "failure") with the probability of success remaining the same every time the experiment is conducted.

4.12.1 The Lesson Excerpt

He presented the class with an experiment that involved rolling a die as depicted in the following lesson excerpt:

Mr McLaren: Let's consider the probability of rolling the dice and getting the number 2. The probability of success is 1/6 and the probability of failure is 5/6. What we are looking for we define as success and the rest we define as a failure. Say we are going to roll the die 4 times. So, what are the chances are we will get no 2's, one 2, three 2's or four 2's?

Mr McLaren drew up a table and recorded FFFF on the board indicating four failures and clarified that the probability of "not obtaining any 2's" is $\left(\frac{5}{6}\right)^4$. He then introduced the parameter p and its complement q, to represent success and failure respectively. Mr McLaren continued to fill in the table (see Figure 4.30), guiding the students along the way with questions such as "I've got all of the different combinations there so, what am I going to write this time?" As he was completing the case for the occurrence of three 2's, Kale suddenly exclaimed "It's Pascal's Triangle!"

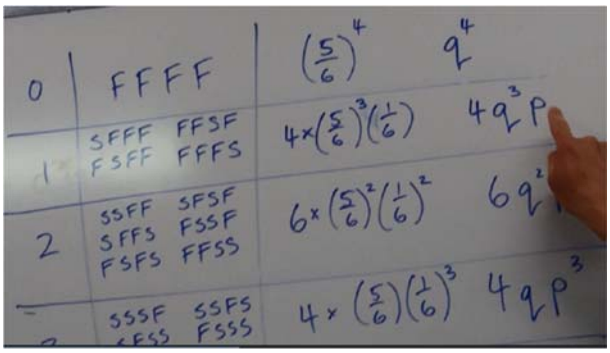


Figure 4.30. Mr McLaren's probability table depicting sample space for rolling 2 for four tosses of the die

Mr McLaren nodded enthusiastically as he continued to fill in the table and then turned to Kale with a smile and said "You're on the ball! Yes, Kale has rightly pointed out, that it looks like Pascal's triangle or the?" He paused to give the students the opportunity to recognise the connection between the pattern of coefficients, 1 4 6 4 1, shown in Figure 4.31 and the coefficients associated with binomial expansions such as, $(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$. Grace whispered "binomial". Mr McLaren heard her "Yes, the binomial theorem or the expansion of a binomial where the two terms are failure or success. And you can see Pascal's triangle with the 1 4 6 4 1". Mr McLaren pointed to the coefficients in the table as he listed them, and the completed table is shown in Figure 4.31.

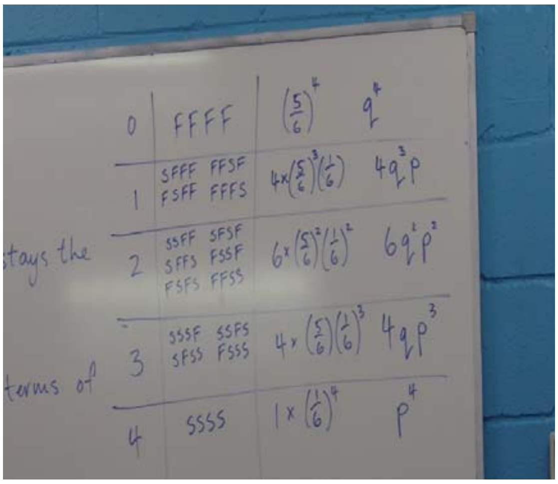


Figure 4.31. Mr McLaren's completed table illuminating the link between the binomial probability distribution and the binomial theorem.

4.12.2 Mr McLaren's Perspective

In his post-lesson interview Mr McLaren elaborated on his instructional decisions in relation to this lesson episode.

Researcher: How did you come to choose the example you used at the start with the rolling of the dice four times.

Mr McLaren: Ah, it's in the text book [see Figure 4.32] but I wanted them to discover it for themselves.

Researcher: What did you want them to discover?

Mr McLaren: Oh, the connection with the binomial theorem and you know Pascal's Triangle. I wanted them to see it and figure out for themselves.

Researcher: They seemed to catch on quite soon.

Mr McLaren: Yes, I was quite pleased.

Researcher: Have you done anything different this year in the way you introduced the topic?

Mr McLaren: Yeah well in the past I might have just led them through the text and tried to still do the discovery thing but I think it works better if it's done on the board and done in that way, rather than using the text as the basis of using it. So, I think leading them through it step by step is better than saying here it is let's have a look at it. It does take longer though but I think it's probably worth it because it gives them that feeling of success before you start or as you lead into a unit.

If X represents a random variable that has a binomial distribution, then it can be expressed as:

$$X \sim \mathrm{Bi}(n, p) \text{ or } X \sim \mathrm{B}(n, p).$$

Translated into words, $X \sim \mathrm{Bi}(n, p)$ means that X follows a binomial distribution with parameters n (the number of trials) and p (the probability of success).

Consider the experiment where a fair die is rolled four times. If X represents the number of times a 3 appears uppermost, then X is a binomial variable. Obtaining a 3 will represent a success and all other values will represent a failure. The die is rolled four times so the number of trials, n, equals 4 and the probability, p, of obtaining a 3 is equal to $\frac{1}{6}$. Using the shorthand notation, $X \sim \mathrm{Bi}(n, p)$ becomes $X \sim \mathrm{Bi}(4, \frac{1}{6})$.

We will now determine the probability of a 3 appearing uppermost 0, 1, 2, 3 and 4 times. Obtaining 3 is defined as a success and is denoted by S. All other numbers are defined as a failure and are denoted by F. The possible outcomes are listed in the table below.

Note that $q = 1 - p$; the probability of a failure.

Occurrence of 3	Possible outcomes	Probability	
0	*FFFF*	$1 \times \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$	q^4
1	*SFFF FSFF FFSF FFFS*	$4 \times \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) = \frac{500}{1296}$	$4q^3p$
2	*SSFF SFSF SFFS FSSF FSFS FFSS*	$6 \times \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^2 = \frac{150}{1296}$	$6q^2p^2$
3	*SSSF SSFS SFSS FSSS*	$4 \times \left(\frac{5}{6}\right) \left(\frac{1}{6}\right)^3 = \frac{20}{1296}$	$4qp^3$
4	*SSSS*	$1 \times \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$	p^4

It is interesting to note that the binomial probability distribution is closely related to the binomial theorem (see far right-hand column). Furthermore, if we examine the coefficients of the terms — that is, 1, 4, 6, 4, 1 — it is evident that they are the entries of Pascal's triangle.

Figure 4.32. Table from Hodgson et al. (2013, p. 516) upon which Mr McLaren's table was based

4.12.3 The Students' Perspective

During the focus-group interview, several students commented on Mr McLaren's approach to introducing the binomial distribution:

Jonti: Yeah, the binomial distribution with that table. I don't know, he just helped you to visualise that there are a set number of ways and it follows Pascal's triangle and you just like multiply it out.

Researcher: Yes, and you also recognised Pascal's Triangle Kale, at what point did you realise that?

Kale: When the second 4 went down I was like "Oh" and then I was thinking there was a kind of pattern and I was sort of thinking because we touched on it on Pascal's Triangle for something else a while ago and it kind of became obvious. I think the way that he kind of sort of left it and didn't say it out loud or tell us. He almost let us discover it for ourselves which kind of imprints it in your mind in a strong kind of way.

Jonti: Yeah you work it out as opposed to being told. I think it's easier to remember it, it sticks there for longer.

Grace: Like we'd done the binomial theorem originally like earlier in the year, and now it's sort of got another use for it, so you can relate our other work to this, it's not just a random section of work anymore.

Kale: I think, just touching on it well this is just a personal thing, but like if you gain acknowledgement for the thing you have just discovered it sort of you feel a bit chuffed, you think yeah I'm not going to forget about this, it makes it a little bit more meaningful than just being in a maths class.

4.12.4 Commentary

The way in which Mr McLaren enabled the pattern of probabilities to unfold as he filled in the table, and his use of funnelling questions to guide the students thinking, were coded as *classroom techniques. Mathematical structure and connections* was also evident because he illuminated the link between the binomial probability distribution and the binomial theorem.

The post-lesson interview revealed that the table generated by Mr McLaren had come from the prescribed textbook. Instead of showing the students the table upfront, however, Mr McLaren chose to present it in a way that he believed would be more effective for the students in terms of fostering mathematical connections. As such, these interview responses were coded as a combination of *mathematical structure and connections*, *use of instructional materials*, and *classroom techniques*.

In addition, Mr McLaren's responses suggest that his teaching approach was underpinned by a belief that it is desirable for students to "discover it for themselves", and therefore may be aligned with *theoretical underpinning of the pedagogy.* Interestingly his rationale for this view tended to focus on student affect rather than on mathematical understanding per se: "it gives them that feeling of success before you start or as you lead into a unit". It is worth noting that a later version of the Chick et al. PCK framework (i.e., Chick & Beswick, 2017) includes *knowledge of student affect*. The time the data for this study were generated, independently coincided with that evolution of the PCK framework.

The focus-group interview responses suggest that several students appreciated how Mr McLaren enabled them to "discover it for themselves" and to make connections between the binomial distribution and the binomial theorem. Kale and Jonti identified the value of being encouraged to build mathematical connections by themselves without simply "being told" from the outset. Their comments were therefore coded as *mathematical structure and connections* and *classroom techniques*. Grace's response was also classified as *mathematical structure and connections* given that she expressed appreciation for being able to make connections with previously studied concepts. Interestingly, Kale emphasised the affective aspect of receiving teacher recognition for making a mathematical connection on his own, and he attributed this to enhancing his future understanding of the idea.

4.13 Scenario 12: Using CAS to Explore the Skewness of the Binomial Probability Distribution

Table 4.12

Information relating to Scenario 12

Date of Lesson	21st September 2015
Teacher	Mr McLaren
Topic	Probability distributions: Exploring skewness of the binomial distribution
Rationale for selecting the data used to construct this scenario	The complex interplay between a teacher's use of technology as a pedagogical tool, his own mathematical knowledge, and students' unexpected responses to funnelling questions.
Elements of PCK	Use of instructional resources Transformation (KQ) Clearly PCK (Chick et al. framework) Knowledge of representation Transformation (KQ) Clearly PCK (Chick et al. framework) Responding to students' ideas Contingency (KQ) Responding to availability/unavailability of resources Contingency (KQ)

This scenario illustrates aspects of a lesson which explored the effect of changing the parameters n (the number of trials) and p (the probability of success) on the graph of the binomial probability function. The class had been introduced to the binomial probability distribution, including the formula, Pr ($X = x$) $C_r^n\ p^x\ q^{(n-x)}$), during their previous lesson. The following lesson excerpt focuses on Mr McLaren and his students' use of CAS technology to investigate the effect of changing p (the probability of success) on binomial distribution graphs when the number of trials *n*, remains fixed.

4.13.1 The Lesson Excerpt

It was common practice for Mr McLaren to demonstrate the use of the CAS calculator by projecting its screen display (via a computer emulator) onto the whiteboard to enable the students to visualise keystrokes and graphical representations. On this occasion, however, he had forgotten his computer and had to "make do" with providing verbal instructions as the students entered the information into their own calculators. Mr McLaren asked his class to "Go into 'Stat' mode please, and we will try to do this as quickly as possible". He drew the students' attention to information on the whiteboard, shown in Figure 4.33, and introduced the task as follows:

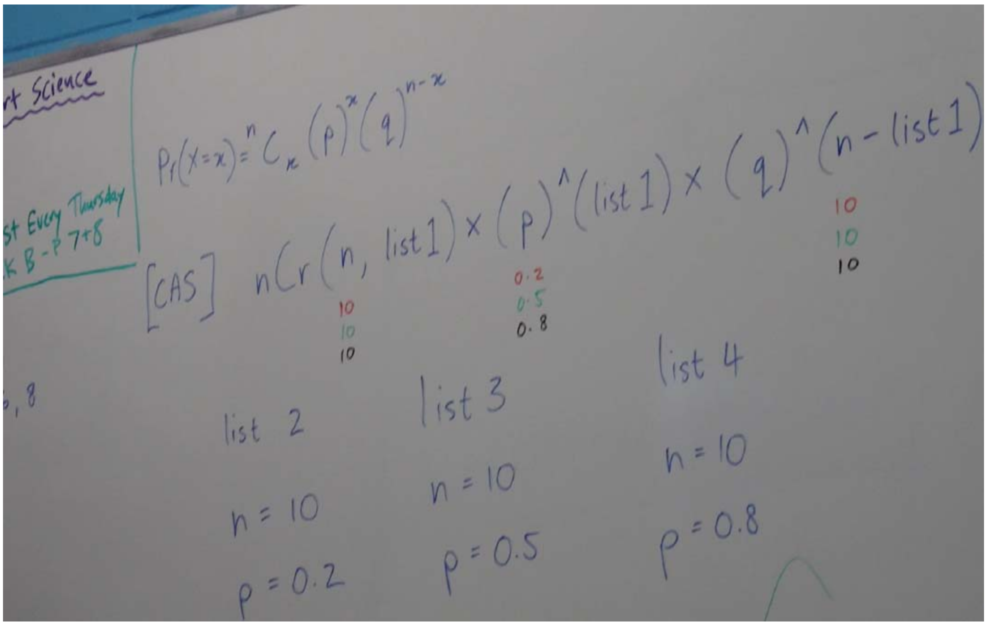


Figure 4.33. Mr McLaren's instructions to be entered using CAS

Mr McLaren: This is what I want you to do. There is the formula for finding probabilities using the binomial distribution [Points to the first formula listed on the whiteboard in Figure 4.33]. I want us to put three things into our calculator [Points to list 2, 3 and 4 in Figure 4.33]. We have to use this [Points to the formula c_r^n (n, List 1) x $p^{List\ 1}$ x $q^{(n-List\ 1)}$]. Can you read that and understand it? So, in List 1 you are going to type the numbers zero through to ten, they will be our *x* values. And in list 2, I want you to type in this same formula [Points again to c_r^n (n, List 1) x $p^{List\ 1}$ x $q^{(n-List\ 1)}$] but with *n* equal to 10 and *p* equal to 0.2.

More than ten minutes passed by and the students were still experiencing difficulty entering the required information. There were many questions and pitfalls as most students were not sure how or where to enter the formulae, and some had difficulty navigating their calculators. Eventually, after considerable trouble-shooting, the students entered the formulae correctly and generated the probability values in lists 2, 3 and 4, as shown in Figure 4.34.

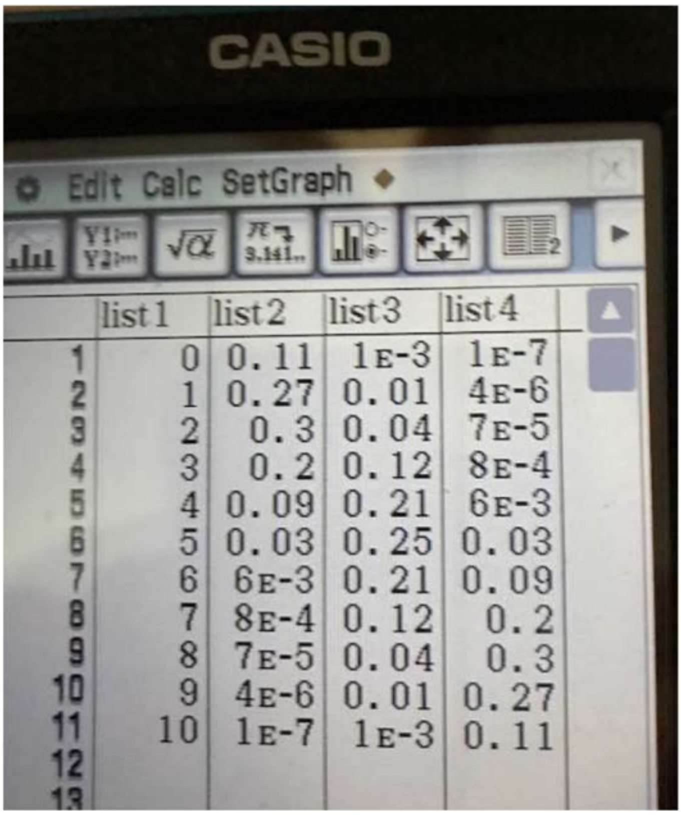


Figure 4.34. CAS screenshot of the three lists showing probability distributions for three different values of *p*

Kale commented on the size of some of the values he had obtained "I'm getting some really small numbers, some miniature numbers!" Mr McLaren, who seemed conscious of how long the students had taken to enter the data, briefly acknowledged Kale's comment with "Yes that's fine" and then called for their attention, "This is taking longer than I was hoping. I've got the three lists here [he showed the students his own calculator screen]. I'll just show you how to get the graphs, so you know that this can be done." Mr McLaren then described the steps involved in obtaining scatterplots for the three distributions (Figure 4.35).

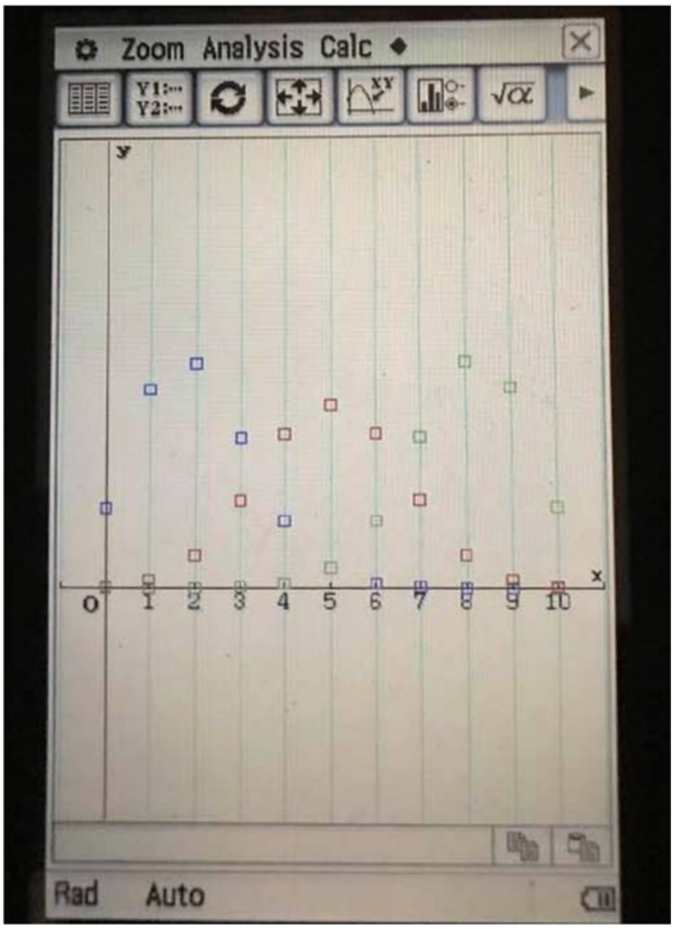


Figure 4.35. Scatterplots of the three lists shown in Figure 4.34

Of the six students in the class that day only one managed to obtain the graphs, so Mr McLaren shared his own calculator and the students formed two groups of three to view the graphs.

Mr McLaren: Now the blue one is graph 1 [List 2 with $p = 0.2$], the red one is graph 2 [List 3 with $p = 0.5$] and the green one is graph 3 [List 4 with $p = 0.8$]. What can you tell me about them?

Kale: Well the red one is symmetrical and the blue one is like the flip of green.

Mr McLaren: You're right yes [He sounds as if he is expecting more information]. What can you tell me about this one? [Points to the red distribution in Figure 4.35].

There was general agreement from the students that the red graph "is symmetrical". Mr McLaren sketched the general shape of the "red" distribution of the whiteboard and probed for further information "What is the graph symmetrical about, what number is it symmetrical about". Eventually, after a student asked what is meant by "symmetrical about", someone suggested that that red distribution was "symmetrical about 5". Mr McLaren also highlighted this as the mean of the distribution. The students were then asked to contemplate the blue graph where p =0.2 (see Figure 4.35). Mr McLaren remarked that "this one [the blue graph] is a bit more challenging to guess what the mean would be".

The class then briefly discussed the green distribution (p = 0.8), including comments such as "it's the opposite of the blue one." Mr McLaren asked the students to consider the similarities and differences between the graphs. One student suggested that the "domains are the same", Kale commented that "the modes are in different places", and Toby noted that "p is different". Mr McLaren pursued Toby's comment by asking what p represents. None of the students however could immediately recall the precise meaning of p. Toby suggested that "p is the probability" but could not elaborate, someone else said "it is the possible outcomes" and Kale asked, "Is it the probability of getting 10?" Mr McLaren shook his head in mock exasperation and said, "Oh dear what happened over the weekend?" and reminded them that "p is the probability of success, and q is the probability of failure". Similarly, the students were not forthcoming with responses when Mr McLaren again probed for students to notice the skewness of the graphs by asking them to explain the *effect* that different values of *p* has on the graph, so the teacher referred to the red distribution (in Figure 4.35) again:

Mr McLaren: Ok this one is nice and symmetrical. What's the probability of success? It's a half, so the probability of failure is also a half. This makes the graph nice and symmetrical about the mean. Now with a lower probability of success [Points to the blue distribution with probability 0.2] it's like we've taken this graph [Points to the red graph] and pulled it to the left. In fact, you will notice that this one [Points to the peak of the blue graph] is actually higher than this one [Points to the peak of the red graph].

Kale suddenly exclaimed "Oh yeah!" and then asked, "So is what the graph is saying is that there is more, there is a lot more chance of getting something lower?" Mr McLaren nodded and responded as follows:

Mr McLaren: Yes, a lower number of outcomes, favourable outcomes, yeah, you're right. So, it's skewing it to a lower number of favourable outcomes. And you know like with a higher number, say the probability that x is 10, it's going to be fairly low with a probability of 0.2 of success. We also see when the probability of success is quite high [Points to the green distribution in Figure 4.35] it has been skewed to the right so these higher values – you would expect that if there was a high likelihood of something occurring, then you would expect to get a lot of them if you keep running the trials.

4.13.2 Mr McLaren's Perspective

During his post-lesson interview Mr McLaren reflected on his efforts to explore the effects of change the parameter p (where n is fixed) on the shape of the binomial distribution as suggested in the following excerpt:

Researcher: Did you do anything different with your teaching approaches to this lesson this year?

Mr McLaren: Probably only because I didn't have my computer I used the calculator which I wasn't particularly happy with, but it did the job.

Researcher: What were you setting out to achieve with the calculator?

Mr McLaren: To demonstrate the effect of changing the probability so if you've got 10 trials and changing the probability from 0.2 to 0.5 to 0.8. I was hoping to be able to put it up on the screen but that didn't happen. I tried to get them to enter it and they weren't very confident at doing it, so it didn't work out as well as I'd hoped but yeah.

4.13.3 The Students' Perspective

The students' focus-group interview and short-reflection responses relating to this lesson, tended to focus on aspects that involved learning how to solve particular problems from the textbook, rather than on exploratory tasks such as the one featured in this Scenario. As it transpired, the students particularly focused on the usefulness of Mr McLaren's demonstration of a problem from the textbook that is featured in the next Scenario. Given that many teaching episodes took place during a given lesson, and the relatively short time-frame available to offer their insights, it is perhaps not surprising that the students prioritised events that related to learning how to do the required mathematics.

4.13.4 Commentary

The lesson excerpt featured in this scenario aligns broadly with *use of instructional materials* given the use of CAS technology as an instructional tool. From a contingency perspective, *responding to the availability/unavailability of resources* was evident when Mr McLaren, in the absence of the CAS emulator, had to adapt by providing verbal instructions for students to follow using their own calculators. His use of the spreadsheet and scatterplots representing the three distributions (see Figures 4.34 and 4.35) was coded as *knowledge of representations*. While some useful comparisons were made between the representations of the three distributions, the extent to which the class could interpret the meaning of the distributions was unclear.

In his post-lesson interview Mr McLaren attributed difficulties relating to the fluency with which the students performed the necessary actions on their calculators, to the absence of the equipment that he normally used to demonstrate CAS applications. Notwithstanding these shortcomings, Mr McLaren's approach to using CAS as a pedagogical tool highlighted challenges associated with recognising opportunities afforded by the technology to illuminate mathematical ideas.

4.14 Scenario 13: "The Tattslotto Problem"

Table 4.13

Information relating to Scenario 13

Date of Lesson	21st September 2015
Teacher	Mr McLaren
Topic	Applications of the binomial probability distribution
Rationale for selecting the data used to construct this scenario	Teacher's use of an alternative representation to assist students to interpret the meaning of a particular representation of a number.
Elements of PCK	Awareness of purpose Foundation (KQ) Teacher demonstration Transformation (KQ) Anticipation of complexity Connection (KQ) Clearly PCK (Chick et al. framework) Deconstructing mathematics into key components Content knowledge in a pedagogy context (Chick et al. framework) Mathematical structure and connections Content knowledge in a pedagogical context (Chick et al. framework) Classroom techniques Pedagogical knowledge in a content context (Chick et al. framework) Teacher insight Contingency (KQ)

This scenario is based on part of a lesson involving the application of the binomial probability distribution. The students had been introduced to the binomial probability distribution in previous lessons and had used it to calculate probabilities. They had also been exposed to the idea that the probability of success may also be expressed as one minus the probability of failure, $(1 - q)$, where q is the probability of failure). The following lesson excerpt focuses on part of a worked solution to the "Tattslotto problem" in Figure 4.36.

In Tattslotto, your chance of winning first division is $\frac{1}{8145060}$. Find the number of games you would need to play if you wanted to ensure a more than 50% chance of winning first division at least once.

Figure 4.36. The Tattslotto problem (condensed from Hodgson, 2013)

4.14.1 The Lesson Excerpt

Mr McLaren began by prompting the students to recognise that for efficiency of calculation, the inequality that describes the probability of winning first division at least once can be expressed as its complement, one minus the probability of never winning (Figure 4.37). It is worth noting that the students had already met, in the previous lesson, the idea that the probability of success is equal to the one minus the probability of failure.

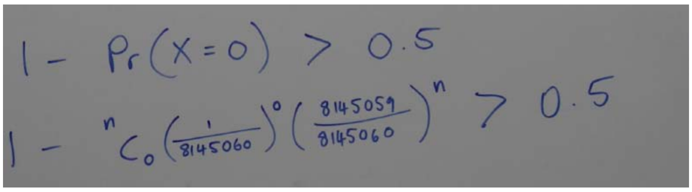


Figure 4.37. Initial stages in determining how many games are required in order to have a 50% chance of winning Tattslotto

After some procedural manipulation, the inequality shown in Figure 4.37 was expressed as: n $\log_e \left(\frac{8145059}{8145060}\right) < \log_e 0.5$. Mr McLaren pointed to the inequality and asked the class "What do we do now?" David suggested dividing both sides of the inequality by $\log_e 0.5$ so Mr McLaren pointed out that "we are trying to get *n* on its own so we need to divide by log of all that [points to $\log_e \left(\frac{8145059}{8145060}\right)$]. Now, is there anything else we need to know about?" There was a pause before Toby tentatively suggested that “The [inequality] sign changes”. Kale quickly retorted “No it doesn’t. I thought you said it only changes when you divide by a negative.” Mr McLaren nodded “That’s right, so why would the inequality change?” “It doesn’t” Kale persisted, looking puzzled. Mr McLaren assured them “It does change, but why?” Someone suggested “because it’s a log” to which the teacher responded, “Yes, well, in a way because it is a log, but why?” David offered “Because there is a rule on our formula sheet?” Mr McLaren shook his head with a smile, “No, there is no rule on your formula sheet”. He paused for a short while and then said “OK, let’s have a look”. Mr McLaren began to write something on the white board but then quickly rubbed it off and changed tack. “OK, let’s think of any log. Now remember the log graph, this is the easiest way to look at it”. He sketched the graph of $y = \log_a x$ as shown in Figure 4.38.

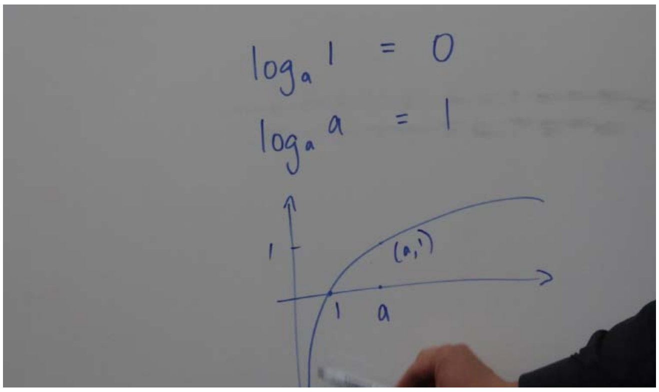


Figure 4.38. Mr McLaren’s sketch of the log graph to show that a particular value is negative

Mr McLaren highlighted the point at x=1 and Toby suddenly exclaimed “Oh, so that’s below one, so it’s a negative, so that’s why you change it around!” Mr McLaren nodded “Good, yes, any value of x less than one, or between zero and one, is negative”. He pointed to the region of the graph between x=0 and x=1 and reiterated that the logarithm to any base of any value for x between zero and one, in this case

$\frac{8145059}{8145060}$, is negative. This was used to explain that when both sides of the inequality are divided by $\log_e\left(\frac{8145059}{8145060}\right)$ the inequality sign changes. “And it’s a good thing too,” Mr McLaren commented as he rubbed the board down, “otherwise we would find that we need to buy less tickets than we would actually need to buy. So, it’s a good thing to look at what you’re actually doing rather than just performing the calculations. OK, can someone evaluate that for me please?” [Points to the right-hand side of the inequality shown in Figure 4.39]

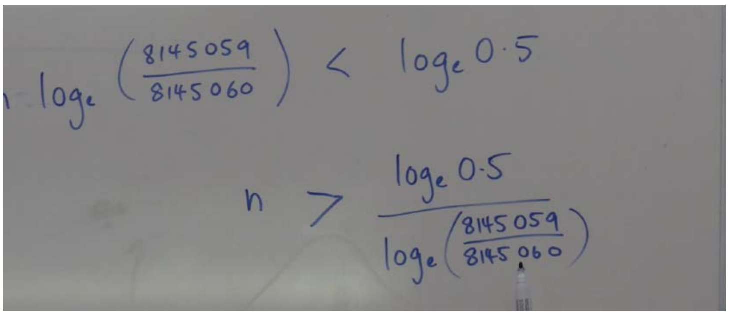


Figure 4.39. Final stages of the calculation of the inequality to determine the number of games

Jonti performed the calculation, yielding 5645727.4. Mr McLaren asked, “Can you buy 0.4 of a ticket?” [The students shook their heads.] “You would still write it to one decimal place, but for your final answer you would round up. You have to round up because if you go less than the 0.4 then you won’t have greater than 50% chance of winning”. Kale looked surprised and called out “So you’d need to buy that many tickets?!” Someone else added “What, just to have a 50% chance of winning once!” Mr McLaren smiled “Yes, so you need to buy a lot.”

4.14.2 Mr McLaren's Perspective

During the post-lesson interview the researcher invited Mr McLaren to elaborate on his decision to draw the graph (of $y = \log_a x$) to illustrate that $log_e\left(\frac{8145059}{8145060}\right)$ has a negative value:

Researcher: The change in the inequality sign generated quite a bit of interest how did you come to decide how to show them why the sign changes?

Mr McLaren: I was just trying to get them well it's a hard one to remember because it [$log_e\left(\frac{8145059}{8145060}\right)$] doesn't look like a negative number but I suppose it strengthens their understanding of logarithms. They were not understanding, well, they hadn't made any connections at that point.

He also commented on his reasons for selecting the Tattslotto problem:

> Probably more so from a non-maths kind of perspective to sort of demonstrate the futility of Tattslotto and the chances of winning that's probably the main reason why I chose that particular question. It wasn't so much a maths choice in that respect.

4.14.3 The Students' Perspective

Several students commented on the usefulness of the way in which Mr McLaren unpacked the solution to the "Tattslotto problem", as indicated in their responses to the researcher's question about the most helpful aspects of the lesson from their own perspectives. This connection between the mathematics itself and the context of the problem seemed to have resonated with Carl in the student focus-group interview:

Carl: Yeah because I was sitting there, and I was like why did you switch it [the inequality sign] because it wasn't dividing by a minus but then it's like no because if you think about it, it's common-sense you're not going to have to only buy a small number of tickets.

Jonti also elaborated, during the focus-group, on the usefulness of Mr McLaren's treatment of the "Tattslotto problem". When asked about the most helpful aspect of the lesson he responded as follows:

Jonti: The log one … [Toby concurs with "The Tattslotto one"]. It was good he kind of like decided on that Tattslotto question because it sort of recaps other things that we knew already so you go through it and refresh your mind on log laws and add the new layer of technicality to it. I don't know, it's just, well, it doesn't look that hard but then the way you've got to go around it with the logs and switching the inequality sign as you go through as well.

Researcher: Did you find anything in the explanation useful in helping you to piece it all together?

Jonti: Yeah I liked how he went through each step, not like skipping over any one of them assuming you would know it. The graph made it a lot clearer as to why you change the sign.

Similarly, David made the following comment in his short-reflection

David: The Tattslotto question was the most useful. It helped me to find the number of games needed for a 50% chance of winning the game and how stupid gambling is.

4.14.4 Commentary

Mr McLaren's worked solution to the "Tattslotto problem" was broadly classified as *teacher demonstration*. The use of funnelling questions to encourage the students to make the connection between the value of the logarithm and the reversal of the inequality sign was coded as *classroom techniques*. When the students did not appear to make this connection by themselves, Mr McLaren sketched the graph of $y = \log_a x$, where "a" represents any base, to assist the students to recognise that the value of $\log_e \left(\frac{8145059}{8145060}\right)$ is negative. This teaching action was coded as *mathematical structure and connections*. *Teacher insight* was also evident given that Mr McLaren appeared to stop and reflect-in-action before changing tack and sketching the graph to provide an alternative representation of $\log_e \left(\frac{8145059}{8145060}\right)$ to enable students to recognise that it has a negative value. He also discussed the reversal of the inequality sign within the context of the "Tattlotto problem", highlighting that it "makes sense because otherwise we would find that we would need to buy less tickets than we would actually need to buy". This aspect of Mr McLaren's knowledge was also identified as *mathematical structure and connections* because he highlighted how the mathematics makes sense within the context of the problem.

During his post-lesson interview Mr McLaren indicated that his decision to draw the graph of $y = \log_a x$ was prompted by his awareness that the students could not readily see that $\log_e \left(\frac{8145059}{8145060}\right)$ has a negative value. The teacher's interview comments were therefore coded as *knowledge of representations*, *deconstructing mathematics into key components*, and *anticipation of complexity.* It would also have been interesting to probe for further information about how Mr McLaren chose the graph as opposed to some other means to illustrate the idea. Interestingly, he also justified his choice of example because it highlighted the very low probability of winning Tattslotto. In this instance, his *knowledge of examples* was linked to *awareness of purpose* because he identified the relevance of the problem to illuminating an idea in the broader real-world context.

The students' perceptions of Mr McLaren's actions support evidence provided in the other data sources. The graph representing the relationship between the value of x and its logarithm was particularly noticed and appreciated by the students (*mathematical structure and connections*). Both *mathematical structure and connections* and *awareness of purpose* were evident in the way in which the students noticed and discussed the connection between the reversal of the inequality sign and the reality of the number of Tattslotto games that would need to be played.

4.15 Scenario 14: The Hospital Problem

Table 4.14

Information relating to Scenario 14

Date of Lesson	18^{th} August 2014
Teacher	Mr Jones
Topic	Discrete probability distributions: Applications of binomial and hypergeometric distributions.
Rationale for selecting the data used to construct this scenario	Teacher's focus on procedures in a situation that also afforded the opportunity to address the conceptual underpinnings of a mathematical idea.
Elements of PCK	<u>Teacher demonstration</u> Transformation (KQ) <u>Classroom technique</u> Pedagogical knowledge in a content context (Chick et al. framework) <u>Concentration on procedures</u> Foundation (KQ)

This scenario is based on aspects of the final of a sequence of lessons on a topic involving the application of discrete probability distributions. Mr Jones began the lesson with an overview of the key differences between the binomial and hypergeometric distributions by referring to a flowchart that he had pre-prepared (Figure 4.40).

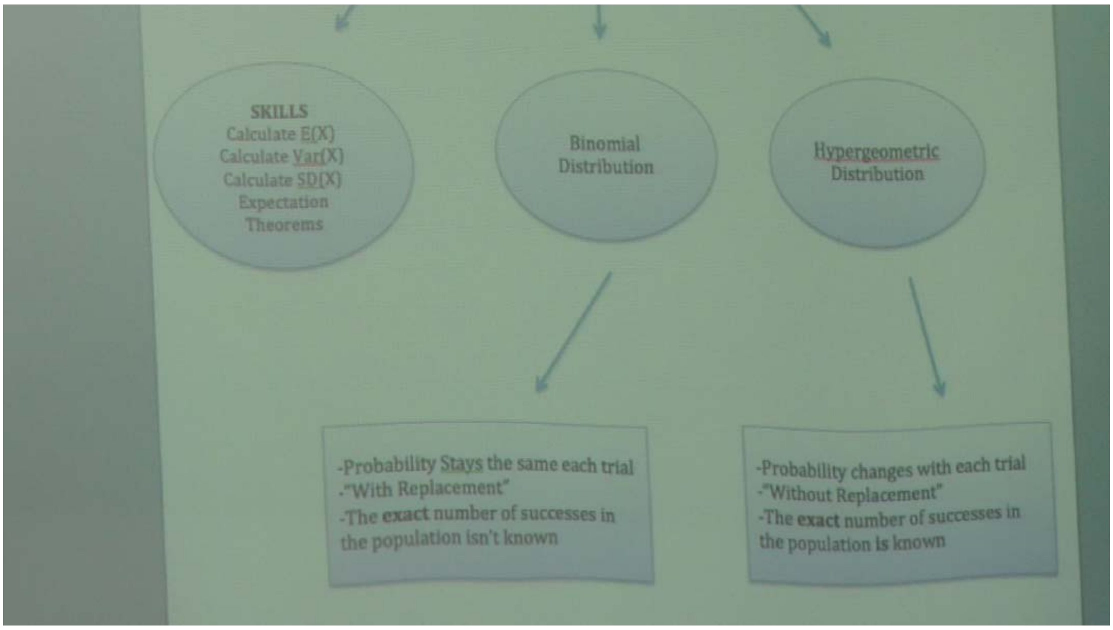


Figure 4.40. Part of Mr Jones flow chart titled 'Probability revision'

Mr Jones then guided the class through worked solutions to each of the items on a worksheet. The worksheet provided a set of revision items on the topic of discrete probability distributions and comprised a range of scenarios that involved either the binomial distribution, or the hypergeometric distribution.

The following lesson excerpt focuses on Mr Jones' worked solution to the following item from the worksheet: *A city has two hospitals. In the larger hospital there is an average of 45 babies born each week, and in the smaller hospital there is an average of 15 babies born each week. In any given week, which of the two hospitals is more likely to have 60% or more of their babies being boys?*

4.15.1 The Lesson Excerpt

Mr Jones: So, we've got two scenarios here. We've got a larger hospital and a smaller hospital. Ok, so we are looking at a typical week. So, is it binomial or is it hyper-geometric? [The students were not forthcoming with a response] OK, let's just take one hospital so we don't confuse things – let's just take the 45 babies, the larger hospital. Alright there are 45 babies born each week, what's the chance of a baby being a girl? [A student responds with "fifty-fifty"] Yes fifty-fifty. In reality it's a tiny bit different but essentially, yes, it's fifty-fifty. So our probability of success is 0.5. Now this is where this is a good question because this 60% [Mr Jones points to the 60% written in the question] can throw people off because some people can latch on to that and think that's the probability. But no, because it's boys or girls our probability is 0.5. Alright so we've got 45 babies born with probability 0.5 [of any baby being a boy]. Now we are looking at 60% being boys, so out of those 45 what are we after there? Ok guys we've got to work out 60% of 45. How do we do that? [Students suggest using their calculator and someone works it out to be 27. Another student points out that it is 27 **or more**]

Mr Jones: Yes 27 or more, now is it binomial or is it hypergeometric?

Alan: It's binomial.

Mr Jones: Why is it binomial?

Alan: Because you don't know how many is in the population.

Mr Jones: OK, so there are two things here. When I did my little flow chart [Figure 4.40] there were two elements to this, knowing how many is in the population, and also whether the probability changes each time. Each time a baby is born what is the probability of getting a boy or a girl, Simon?

Simon: 0.5

Mr Jones: Exactly, so there are a number of ways you can explain this. It can be that it's replaced, [Refers to sampling with replacement] it can be that you don't know the number of successes in the population, and it can be that the probability doesn't change with each trial. You can give lots of different ways of explaining why it is binomial, but to me this is the key one here, the probability doesn't change each time, and that's one of the key features of a binomial distribution. [He then guides the class through the solution using their CAS calculators using the binomial cumulative distribution function to obtain an answer of 0.1163]

Mr Jones: Now, always try and make sense of your answer, we got 0.1163 is that a very high probability? [The students agree that it is not]

Mr Jones: So, let's look at our scenario. We have a situation where it is 50:50 so 60% or more of the one gender being born out of 45, there is not a high chance of that happening. Out of 45 you could expect anywhere between 22, 23, 24, 25 of them to be boys and to be the most likely result. So, to be 27 or more, it's not overly likely. Ok so the small hospital, it's the same deal [Mr Jones lists the relevant information to be entered into the calculator] So what's 60% of 15? 9. OK so has anyone worked it all out to save a bit of time? [One student who has worked out the probability on his CAS calculator says 0.3036] Ok, so obviously it is more likely to be the small hospital. [Mr Jones then immediately moves on to look at the next question on the worksheet]

4.15.2 Mr Jones' Perspective

During his post-lesson interview, Mr Jones explained his intention to make the lesson a review of discrete probability distributions with particular focus on selecting the appropriate distribution: "So if I was writing up my lesson in some kind of formal way my objective for today would be to draw it all together – from the full overview to then just concentrating on identifying when it is binomial or hyper-geometric or neither." In relation to the "hospital problem" itself, the researcher asked Mr Jones if he had considered asking the students to first predict which hospital they thought may have the higher probability.

Mr Jones: Well what occurred to me last night when I was preparing this [lesson], and I didn't talk about it today because I didn't think I was across it myself enough, is could you have predicted which is the – could you have predicted which hospital – I mean I could have done that prior to doing the question or as a reflection after – you know why did the small hospital have more than double the probability of the previous one yeah. That is something as a teacher I need to keep in mind it's one thing to churn out answers which is fine in one sense but also sometimes say oh look at that, I wonder why, because I mean I'm surprised some of the kids didn't say "Oh gee, you know, why are the probabilities so different with the small and large hospitals". I reckon if I'd asked them beforehand, they would have thought "Oh one might be a bit bigger than the other but there won't be much difference because the scenarios are the same". If I had more time in this course and I'm not making lame excuses here, but I'd do far more of that let's stop and look at this a bit further let's look at this a bit deeper.

He also reflected more generally on the tension associated with prioritising those aspects of the content which should be the key focus of his teaching of the topic of probability:

Mr Jones: I suppose the development [derivation] of the binomial equation and hypergeometric is something that you know we haven't really done either. I suppose it gives a fuller understanding of what they are doing, you know it goes back to understanding, we're not just plucking these formulas out of the air you know. But that doesn't mean there is no understanding because the focus of the understanding comes from like what we did today with identifying which distribution to use – even though you're probably not developing it right through every facet, but you are focussing on getting to the end product. Maybe that's what is important in the sense that out in the real world it's about what tool I am going to use.

Mr Jones also commented on the common difficulties that students have in relation to the topic of probability within the context of Mathematics Methods.

Mr Jones: One, I think the difficulties will come when I mix all the probability types up. Not so much the normal distribution, normal is sort of fairly obvious but it's the binomial and hyper-geometrics. Because sometimes, OK it will overtly say with replacement or without replacement and that's OK. But the real tricky ones are, there will be questions like "statistics has shown or history has shown that 10% of the population are whatever OK, in a population of 400 people so that one is actually binomial because you don't actually know that in that group of 400 people the population N, you don't know that 10% of them are whatever they are, you know, got asthma or something for example ok, you don't know. But if the question had said out of a group of 400 people 40 are known asthmatics, then you use hyper-geometric because you actually know how many successes there are in the population. That's a subtle thing but that's something I'll have to go through with them.

4.15.3 The Students' Perspective

During their post-lesson focus group, several students commented on the most useful aspects of the lesson from their perspective.

Simon: Probably when Mr Jones explained how to, he just went through the set of questions there where it was all binomial, hyper-geometric or neither. I didn't really understand the difference between them until we did that. Like I knew one of them was with replacement and one of them was without, but now I've got two or three different ways I can distinguish which one's which. That helped me a lot.

Alan: Yeah just the difference between binomial and hyper-geometric yeah because I was a bit confused before.

Simon: I think it probably helped as well that he was actually doing the questions on the board instead of just asking us which one it was and then going on to the next one. So he actually showed us how to work each one out. I think that's what made it click for me after the second or third one I started to know which one was which.

Danny: When the flow chart [Figure 4.40] was up there and it had the two headings binomial and hyper-geometric then it said underneath it what you had to look for to try and distinguish which-was-which, and then we did the questions and you could see what you had to look for in the questions.

Similarly, Alan provided the following response in his short reflection.

Alan: The most helpful thing was him [Mr Jones] going through the differences between binomial and hypergeometric distributions by doing questions as a class. This helped to identify which distribution the questions are asking about. This makes it faster to complete questions and use the right formula.

4.15.4 Commentary

Mr Jones' step-by-step worked solution to the "hospital problem" was coded as *teacher demonstration*. His use of funnelling questions and the flowchart to guide the students to determine the type of probability distribution, both generic teaching techniques, were identified as *classroom techniques*. The researcher was aware that versions of the hospital problem have been used in research (e.g., Kahneman & Tversky, 1972; Watson, 2000) to explore peoples' understanding of sampling and variation. Influenced by this literature, the researcher saw the "the hospital problem" as an opportunity for Mr Jones to address the key probabilistic idea that greater variation is more likely to occur in small samples – hence the small hospital is more likely to record more days with 60% or more boys born. Although the teacher raised the idea that "it's not overly likely" that 27 out of the 45 babies born in the hospital in a given week would be boys, he did not elaborate or attempt to facilitate a discussion about why the probabilities obtained for the two hospitals were markedly different. Rather, Mr Jones' demonstration focused primarily on the processes involved in calculating the two probabilities and was therefore coded as *concentration on procedures*.

The post-lesson interview with Mr Jones offered useful perspective on his instructional decisions based on the lesson, including his approach to the "hospital problem". *Concentration of procedures* was demonstrated when the teacher articulated that the purpose of his lesson was to provide a review of binomial and hyper-geometric distributions, and how to recognise and apply the relevant distribution to specific scenarios. By his own admission, Mr Jones claimed not to be "across it" enough himself to have led a discussion about why the probabilities for the two hospitals were so different. His reasons for avoiding focusing on mathematical ideas in depth focused on time constraints associated with covering the content of the Mathematics Methods course and was therefore coded as *knowledge of curriculum*. In addition, Mr Jones justified his limited focus on developing students' conceptual understanding of the formulae for the distributions (binomial and hyper-geometric) by suggesting that learning how to recognise and apply the correct distribution to solving real world problems is a key priority. This aspect of Mr Jones' interview was also linked to *concentration on procedures*. In addition, *theoretical underpinning of the pedagogy* was relevant because his response alluded to his beliefs about what important features of understanding mathematical ideas.

Data from the focus group interviews and short reflections suggest that several students found Mr Jones' worked solutions particularly useful from their perspective (*teacher demonstration*). Specifically, Alan and Simon highlighted the ways in which Mr Jones' demonstrations helped them to recognise and select the appropriate probability distribution relevant to a given problem. Danny's interview comment highlighted the usefulness of Mr Jones' flowchart (*classroom techniques*) in helping him to determine the correct distribution to use.

4.16 Summary of the Chapter

This chapter presented the results in the form of Scenarios, each of which included a specific lesson episode depicted from the researcher's perspective, followed by corresponding insights from the viewpoints of the teacher and the students. Collectively the scenarios provide a detailed portrayal of the enactment of PCK in the senior secondary mathematics classrooms featured in this study.

The scenarios were structured in a way that facilitated the analysis of PCK in a multi-faceted way, through a broad range of filters provided by the KQ and the PCK framework. In particular the scenarios allowed inferences to be drawn in relation to the three research questions, which focus on PCK from the three different perspectives (researcher, teacher, and student). The next chapter responds to the research questions by drawing upon the findings presented in this chapter.

Chapter 5

Discussion

The previous chapter presented detailed snapshots of the PCK evident in the lessons from the perspectives of the researcher, the teachers, and their students. This chapter addresses the three research questions by drawing upon and analysing these multiple sources of evidence of PCK. The chapter concludes with a summary that brings together and discusses the key findings from each of the three perspectives.

This study explored mathematics PCK at the senior secondary level by focusing on the following three research questions.

1. What aspects of mathematical pedagogical content knowledge are evident in the interactions between teachers and their students during the teaching and learning of senior secondary mathematics content?
2. What aspects of mathematical pedagogical content knowledge do teachers discuss and attribute their instructional decisions to when analysing their interactions with students during the teaching and learning of senior secondary mathematics content?
3. What aspects of mathematical pedagogical content knowledge are identified by students as having an impact on their learning of senior secondary mathematics content?

These questions were designed to allow an exploration of PCK from multiple perspectives and data sources.

5.1 Overview of Aspects of Mathematical Pedagogical Content Knowledge

This section discusses the multiple and often intertwined elements of PCK evident from the perspectives of the researcher, the teachers, and their students by addressing the three research questions. These elements are presented under the four dimensions of the Rowland and colleagues' KQ (e.g., 2005): *foundation*, *transformation*, *connection*, and *contingency*. The four dimensions provided a useful framework upon which to examine the different ways in which PCK came into play in the classroom, and/or influenced the teachers' instructional decisions. *Foundation* is concerned with the PCK held by a teacher regardless of whether it is implemented in the classroom (e.g., knowledge of common errors that students make). *Transformation* comprises elements of PCK that focus on transforming content in accessible ways for the learner, such as the use of demonstrations and representations. *Connection* includes PCK that foregrounds the inherent connections between mathematical ideas and the identification of features that affect the complexity of these ideas. *Contingency* encompasses the PCK that teachers draw upon in their moment-by-moment instructional decisions, and is considered by some scholars (e.g., Mason & Davis, 2013; Mason & Spence, 1999) as a particularly powerful and nuanced form of PCK.

The broad range of elements of PCK evident in the data generated in this study support later conceptualisations of PCK that build upon the work of Shulman, including Rowland et al. (2005), Magnusson et al. (1999) and Chick et al. (2006).

These later conceptualisations, as emphasised by Graeber and Tirosh (2008), have widened the definition of PCK to include a broader range of elements. As such, a combination of components of the KQ and the Chick et al. framework provided an extensive set of filters through which to explore the multiple elements of PCK evident in this study.

The presence of elements that were clearly pedagogical content knowledge but that were not explicitly referred to in the KQ led the researcher to modify the four original dimensions of the KQ to incorporate those elements for the purposes of this study. For example, *knowledge of curriculum* and *knowledge of assessment* from the Chick et al. PCK framework were placed within the foundation dimension. Like the original contributory codes of the foundation dimension, *knowledge of assessment* is held by the teacher irrespective of its implementation in the classroom. In addition, *mathematical structure and connections* and *deconstructing mathematics into key components* were included in the connection dimension because those elements aligned with Rowland and colleagues' (2009) idea that the connection dimension is concerned with illuminating the coherence among seemingly separate aspects of mathematics content. The researcher's decision to use *mathematical structure and connections* and *deconstructing mathematics into key components* instead of the similar connections-related elements from the KQ, is discussed in the Methodology chapter in Section 3.9.2. *Classroom techniques* was also placed within the connection dimension because the instances of *classroom techniques* observed by the researcher specifically related to helping students make mathematical connections. It is important to emphasise, however, that even though the classroom techniques observed in this study related to ways in which the teachers made mathematical connections, *classroom techniques*, as a category of teacher knowledge, is not inherently associated with any one dimension of the KQ. Given its generic nature, it is possible to envisage *classroom techniques* being associated with any of the four dimensions of the KQ depending on the focus or purpose of the technique.

In keeping with the structure of the research questions, the following three sections examine evidence of PCK from the three different perspectives (i.e., researcher, teacher, and student). Many of the scenarios featured in these sections are addressed on more than one occasion, under the filters of different PCK elements and from different perspectives. This overlap reflects the dynamic and interconnected nature of PCK, a view that is supported by several scholars (e.g., Hashweh, 2005; Fennema & Franke, 1992; Magnusson et al., 1998).

5.2 Research Question 1: PCK Enactment in the Classroom as Viewed by the Researcher

The first research question was addressed using data generated from the observations and video recordings of the teaching and learning interactions between the teachers and their students during the instructional phases of the 18 lessons. This subsection discusses those elements of PCK that were particularly evident from the researcher's observations of the lessons captured through video footage. Illustrative examples of these elements of PCK, drawn from the scenarios in Chapter 4, are discussed under the dimensions of foundation, transformation, connection and contingency.

5.2.1 Foundation

Several elements from the foundation dimension were observed by the researcher: *concentration on procedures*, *knowledge of curriculum*, *knowledge of* *assessment*, *adherence to textbooks*, *use of mathematical terminology*, and *overt display of subject matter knowledge*.

Concentration on procedures was broadly evident in the researcher's observations of the teachers' step-by-step worked solutions to textbook problems. For example, in Scenarios 1 ("Tom's suggestion"), 2 ("The particle problem") and 14 ("The hospital problem") the teacher focused primarily on the procedures involved in solving specific problems. In some cases, *concentration on procedures* was linked to the teachers' *knowledge of curriculum*, as discussed further on.

Knowledge of curriculum, an element of the Chick et al. PCK framework, refers to a teacher's knowledge of the scope and sequence of the content of mathematics curricula. In this study, the teachers demonstrated *knowledge of curriculum* when they discussed the nature and content specific to the Mathematics Methods course. For example, in Scenario 10 ("Introduction to variance") Mr McLaren explained to his students that they did not need to address the proof of the variance formula because it was not a requirement of the course. In this case, his *concentration on procedures* was linked to his *knowledge of curriculum*. More extensive evidence of *knowledge of curriculum* is provided, from the teachers' perspective, in Section 5.1.2.

Like *knowledge of curriculum*, for the purposes of the study *knowledge of assessment* was included within the foundation dimension. *Knowledge of assessment*, an element of the Chick et al. PCK framework, is described by the authors as evident when a teacher discusses or designs tasks and activities to assess learning outcomes (e.g., Chick & Beswick, 2017). This broad definition implies that the teacher has the role of decision-maker in relation to the nature and design of activities to assess student achievement. In this study, *knowledge of assessment* was observed to play out when the teachers discussed the expectations of the external examination with their students. For example, in Scenario 8 ("Trig pops up everywhere") Mr Jones discussed an example (i.e., $\int_{k}^{\pi} \cos(2x)\, dx$ = - $\frac{\sqrt{3}}{4}$, find the value of k given that $0 < \; k < \frac{\pi}{2}$) in terms of its suitability as an examination-type question. He told his students that questions that involve "finding *k* somewhere" often appear in the final examination and are "worth quite a few marks". Similarly, in Scenario 1 ("Tom's suggestion") Mr Jones made a point of using standard algebraic techniques to solve a specific optimisation problem because he believed the problem was well-suited to the technology-free section of the external examination given its "nice neat" exact value solution. These instances typify the researcher's observations of the teachers' enacted *knowledge of assessment* which was limited to the context of the external examination, even though this element potentially encompasses knowledge of broader principles of assessment.

The other three elements from the foundation dimension that were observed by the researcher – *adherence to textbooks*, *use of mathematical terminology*, and *overt display of subject matter knowledge* – originate from the foundation dimension of the KQ. There were, however, subtleties and nuances in the way in which these elements were used to code the data generated in this study, because the use of textbooks, mathematical notation, and the display of subject matter were common, often unremarkable characteristics of the senior secondary mathematics classroom. As such, *adherence to textbooks*, *use of mathematical terminology*, and *overt display of subject matter knowledge* were assigned to teaching and learning instances that were particularly notable.

Adherence to textbooks was evident in Scenario 4 ("Anti-differentiation with Mr Taylor) when the teacher urged his students to conform to a convention used in the prescribed textbook (and the broader course context) that involved distinguishing between finding *an* antiderivative and finding *the* antiderivative. In the case of the former, the arbitrary constant of integration must be set to zero. While Mr Taylor acknowledged and supported the students' suggestions that "an" arbitrary constant could be "any" arbitrary constant, his *adherence to textbooks* was related to his awareness of when it is appropriate to conform to the convention adopted by the textbook.

Also, of interest within the context of Scenario 4 ("Anti-differentiation with Mr Taylor") was the teacher's *use of mathematical terminology*. While the use of mathematical notation is central to teaching senior secondary mathematics, there were some instances in this study where the teacher attended to this notation in particularly explicit ways, such as Mr Taylor's special focus on the correct use of integral notation. The teacher offered an interesting justification for this emphasis on mathematical notation as addressed in Section 5.1.2.

Like the *use of mathematical terminology* and *adherence to textbooks*, another element of the foundation dimension, *overt display of subject matter knowledge* was used in this study to identify the extraordinary within the ordinary. The teachers demonstrated strong knowledge of the content of the Mathematics Methods course, particularly in relation to solving typical textbook and examination-type problems. In this sense a teacher's use of subject matter knowledge is part of the landscape of a senior secondary mathematics classroom. *Overt display of subject matter knowledge*, however, is evident when a teacher draws upon his/her content knowledge in ways that transcend the knowledge required for performing standard solution methods and explanations. For instance, in Scenario 10 ("Introduction to variance") Mr McLaren attempted to show algebraically why the "b" disappears in the relationship $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$. His *overt display of subject matter knowledge* was related to his awareness of the relationship between $\mathrm{Var}(aX+b) = \mathrm{E}(aX + b)^2 - [E(aX + b)]^2$ and the relevant expansion process. Similarly, in Scenario 13 ("The Tattslotto problem") Mr McLaren called upon his own content knowledge to draw the graph of $y = \log_a x$ to prompt the students to see that a particular logarithm had a negative value. While the problem itself did not require the graph as part of its solution, Mr McLaren used it to assist the students to make sense of a crucial step in the solution process that pertained to recognising that $\log_e\left(\frac{8145059}{8145060}\right)$ is negative.

5.2.2 Transformation

Teacher demonstration, *knowledge of examples*, *knowledge of representations*, and *use of instructional resources* were key elements of PCK from the transformation dimension, that were observed by the researcher during the instructional phases of the lessons.

Teacher demonstration was a broad and prominent aspect of the teachers' enacted PCK, which is not surprising given that direct instruction (e.g., Kivunja, 2015) was the teachers' dominant professional practice. The instructional phases of each lesson involved the teacher modelling mathematical procedures and solutions to text-book examples. Rowland et al. (2009) point out that teacher demonstrations are typically accompanied by a commentary from the teacher explaining how or why a specific process works and drawing attention to potential challenges and pitfalls; these were common aspects of the performance of the teachers in the present study.

For example, in Scenario 1 ("Tom's suggestion") Mr Jones demonstrated the solution to an optimisation problem by providing detailed step-by-step explanations of the algebraic processes involved in calculating the required minimum value. Other demonstrations, including those depicted in Scenarios 8 and 9 ("Trig pops up everywhere" and "e^2 is just a number', respectively), focused on drawing students' attention to possible difficulties or pitfalls in the particular problem. In this sense, *teacher demonstration* was closely linked to the teachers' knowledge of the examples they chose to demonstrate.

Knowledge of examples was a substantial component of the teachers' enacted PCK. Examples were used to introduce mathematical ideas, to demonstrate worked solutions, and to assign text-book exercises for students to complete in class by themselves. As described by Zodik and Zaslavsky (2008), instructional examples represent specific cases from which broader generalisations can be made. For instance, Scenario 10 ("Introduction to variance") illustrates the way in which Mr McLaren used the graphs of two strategically chosen probability distributions to introduce the concept of variance.

Knowledge of examples was also evident in the tactical ways in which the teachers used examples. In Scenario 5 ("Anti-differentiation with Mr Taylor") the teacher introduced his students to the special-case integral $\int \frac{1}{x}\, dx$ by presenting them with $\int 5\, x^{-1}\mathrm{d}x$, immediately following a sequence of examples that involved the anti-differentiation of expressions of the form ax^n (where $n \neq -1$). In other words, the teacher invoked a situation of cognitive conflict by letting the students attempt to use the "power rule" to anti-differentiate $\int 5\, x^{-1}\mathrm{d}x$, only to find that the rule did not apply. Mr Taylor's use of $\int 5\ x^{-1}dx$ served as a counter example (Zazkis & Chernoff, 2006) to refute the idea that the power rule works in all situations.

The ways in which the teachers sequenced examples for demonstration was also of interest. In Scenario 5 ("Anti-differentiation with Mr Taylor") the teacher decided on the order of a set of examples, involving the anti-differentiation of expressions, according to their procedural complexity. Later in his post-lesson interview, Mr Taylor elaborated this teaching decision as highlighted in more detail in Section 5.1.2.

The teachers' use of representations was also evident in their use of examples. *Knowledge of representations* is concerned with teachers' awareness of the multiple ways of presenting mathematics content, including the use of physical, diagrammatical, and conceptual representations (Leinhardt, Putnam, Stein, & Baxter, 1991). The teachers used representations as a routine part of their teaching because diagrams, graphs, and tables were inherent in the Mathematics Methods course content. Of interest, from a PCK perspective, were the ways in which the teachers selected and used representations to transform the content for their students. In Scenario 10 ("Introduction to variance") Mr McLaren demonstrated *knowledge of representations* in his construction of two data sets, and their graphs, to draw attention to the idea of variance. *Knowledge of representations* was also demonstrated by Mr McLaren in Scenario 13 ("Tattslotto problem") when he chose to sketch the general graph of $y = \log_a x$ to prompt the students to realise that $\log_e\left(\frac{8145059}{8145060}\right)$ has a negative value. Mr McLaren's students particularly noticed his use of representation in this instance, as discussed further on in Section 5.1.3. The instances of *knowledge of representations* featured in Scenarios 10 ("Introduction to variance") and 13

(“Tattslotto problem”) highlight how the teacher drew upon his own mathematical content knowledge to represent the content in useful ways for the students. These findings support Leinhardt and her colleagues’ (1991) idea that teachers’ capacity to use multiple representations in pedagogically powerful ways is influenced by the depth of their own mathematical content knowledge.

It was also interesting to consider the teachers’ reasons for choosing specific representations. For instance, in Scenario 1 (“Tom’s suggestion”), Mr Jones demonstrated the solution to an optimisation problem using standard differentiation approaches that involved algebraic representations. The relevance of *knowledge of representations* in this case was not so much to do with the use of algebraic representations per se, but rather the absence of the inclusion of a graphical representation of the solution. A supporting visual representation of the minimum value obtained could also have been demonstrated using the CAS calculator. Mr Jones’ focus on standard algebraic techniques alone, was based on the idea that the problem was suited to the technology-free section of the examination, a point he had emphasised to the students during the lesson. In this sense, Mr Jones’ *knowledge of representations* was mediated by his perception of the expectations of the Mathematics Methods course, that is, by his *knowledge of assessment*.

Another aspect of PCK that was often linked to *knowledge of representations* is *knowledge of instructional resources*. The main instructional resources used by the teachers were the prescribed textbook (Hodgson et al., 2013) and the CAS calculator, and on rare occasions, other computer technologies (e.g., *Excel* spreadsheets). In this study, the CAS calculator was used as both an instructional resource, and as a functional tool to solve specific types of problems. The functional and pedagogical uses of CAS align with the Mathematics Methods course recommendations that students “should have access to graphics calculators and become proficient in their use” and that “Graphics calculators can be used in all aspects of this course in the development of concepts and as a tool for solving problems” (TQA, 2014, p. 3).

The teachers’ functional use of CAS (e.g., Kendal & Stacey, 2001) was linked to how the technology was used to solve problems that were considered relevant to the technology-enabled part of the examination. For example, in Scenario 7 (“Using CAS for integral calculus”), Mr Jones’ step-by-step demonstration of the keystrokes involved in solving a specific problem, was typical of the functional way in which CAS was used to solve “technology-enabled” problems.

The teachers’ pedagogical use of CAS (e.g., Kendal & Stacey, 2001) was evident in situations such as Scenario 12 (“Using CAS to explore skewness”) when Mr McLaren used the technology to investigate skewness of the binomial distribution. The students, however, experienced difficulties with entering the appropriate information into their calculators and, when analysing the information, did not readily grasp the idea of skewed distributions. In addition, Mr McLaren did not capitalise on the opportunities afforded by the technology to explore the distributions by, for example, drawing connections between the probability values in the spreadsheet lists and the corresponding graphs. This example of a teacher’s attempt to use CAS to explore a mathematical idea, rather than as a functional tool to calculate an answer, exemplifies the complex and multi-faceted nature of teacher knowledge at the senior secondary level. Geiger and his colleagues (2010) point out that the teacher must simultaneously possess the mathematical content knowledge, technological expertise, and the confidence to develop students’ mathematical knowledge.

5.2.3 Connection

The connection dimension comprises elements of PCK that focus on the inherent coherence (e.g., Rowland et al., 2009) and relationships among mathematics ideas. Multiple elements of PCK from this dimension were observed from the researcher's perspective: *anticipation of complexity*, *mathematical structure and connections*, *deconstructing mathematics into key components*, and *classroom techniques*.

Anticipation of complexity related to the teachers' awareness of the cognitive demand of mathematical tasks, as well as the ways in which the complexity was handled within the teaching context. Teachers' understanding of what makes the learning of specific content easy or difficult is a key component of Shulman's original conceptualisation of PCK (1986). The following examples feature evidence of *anticipation of complexity* observed from the researcher's perspective. Additional insight into this element of PCK was provided from the teachers' perspective as discussed in Section 5.1.2.

Anticipation of complexity was evident in Scenario 8 ("Trig pops up everywhere") when Mr Jones identified an example as being potentially challenging for students because it required them to draw upon their knowledge of two arguably challenging topics, one of which the class had studied much earlier in the school year. Mr Jones responded to the cognitive demand of the task by guiding his students step-by-step through the solution process, focusing explicitly on those aspects of the task that were likely to present obstacles for the students. He also encouraged the student to use their official "formula sheet" if they felt they may have forgotten key facts (e.g., whether $\sin 2\pi$ is equal to zero or one). Ultimately, Mr Jones reduced the complexity of the task for the students by providing them with detailed practical steps on how to navigate their way through the problem, particularly if they were faced with a similar example in the examination.

A contrasting example of *anticipation of complexity* was demonstrated in Scenario 9 ("e^2 is just a number') when Mr McLaren anticipated that the inclusion of e^2 as one of the terms of an exponential expression, enhanced its complexity because of students' tendency to interpret e^2 as a variable instead of a number. Mr McLaren addressed this difficulty by allowing time for the students to think through the process of anti-differentiating the expression and letting them fall into the trap of attempting to anti-differentiate the term e^2, before recognising their own error.

From the researcher's perspective, the teachers managed the complexity of mathematical tasks by either reducing the cognitive demand of the tasks, or by supporting the students to confront the complexity. One of the ways in which the teachers encouraged their students to make sense of or to confront challenging aspects of mathematical tasks was through questioning. Teacher questioning is a common generic classroom technique that engenders PCK when used in a specific content context (Marks, 1999). In this study, teacher questioning was classified as *classroom techniques*, an element of the pedagogical knowledge in a content context category of the Chick et al. PCK framework. In Scenario 10 ("Introduction to Variance") Mr McLaren posed funnelling questions (Bauersfeld, 1995) to encourage his students to notice and attend to the different spreads of the graphs of two datasets to introduce the concept of variance. When the students were not forthcoming with the expected responses, the teacher explained the features of the graphs that he had intended for the students to notice in the first place. Mason (1998) points to the challenges involved in the effective use of questioning in the teaching and learning of mathematics, by highlighting the mismatch arising between the responses given by the students and those expected by the teacher. Such challenges are part of the complexity involved in helping students to recognise and appreciate salient connections among mathematical ideas.

As discussed in Chapter 3 Section 3.9.2, *Mathematical structure and connections*, a component of the Chick et al. PCK framework, provided a useful filter through which to examine the nature of the mathematics connections attended to by the teachers. *Mathematical structure and connections* was particularly evident in relation to the teachers' focus on making connections between algebraic processes. In Scenario 5 ("Anti-differentiation with Mr Taylor") the teacher emphasised the connection between the processes of differentiation and anti-differentiation by introducing, and continuing to emphasise, the latter as the process of "undoing" differentiation. This focus on broader structures and generalisations was also evident in Scenario 6 ("It's not a perfect square") when Mr Jones helped his students to recognise that the difference of two squares method of factorisation generalises beyond the familiar situation involving square numbers.

There were, however, instances where the teachers overlooked arguably clear opportunities to address specific connections. These instances were examined in terms of an *absence* of *mathematical structure and connections*. For example, in Scenario 10 ("Introduction to variance") Mr McLaren made the point of telling his students, during the lesson, that they did not need to address the proof of the formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ because the course did not require them to do so. Instead the teacher encouraged the students to focus on the application of the formula given that this was the key focus of the course. In this sense, the absence of *mathematical structure and connections* was linked to Mr McLaren's knowledge of the expectations of the Mathematics Methods course, and therefore linked to his *knowledge of curriculum* and *concentration on procedures*, as discussed under the foundation dimension.

Another situation that involved an absence of *mathematical structure and connections* was illustrated in Scenario 14 ("The hospital problem"), Mr Jones overlooked the opportunity to address key probabilistic ideas that were afforded by the problem. Tensions related to deciding whether to deviate from the lesson agenda by delving into additional mathematical ideas, are discussed within the contingency dimension and from Mr Jones' perspective in Section 5.1.2.

Related to the idea of *mathematical structure and connections* is *deconstructing mathematics into key components*, which also derives from the Chick et al. PCK framework. *Deconstructing mathematics into key components* draws on the work of Ball (2000) and is concerned with unravelling mathematics content in ways that make it visible to the learner. The kind of deconstruction observed by the researcher often focused on unpacking the information required to solve typical textbook problems. For example, in Scenario 1 ("Tom's suggestion") Mr Jones deconstructed a standard optimisation problem into the key processes required to solve it (e.g., devise the function, differentiate the function, set the derivative equal to zero).

In some instances, a teacher attempted to deconstruct mathematics content in ways that deviated from standard solution processes. For example, in scenario 10 ("Introduction to variance") Mr McLaren endeavoured to deconstruct the formula Var $(aX + b) = a^2 \text{Var}(X)$ in response to a student's question about "what happened to the *b*?”. The teacher’s explanation, however, lacked clarity, likely because he was not usually called upon to unpack the formula. This instance is also discussed in terms of contingency further on.

These examples of *deconstructing mathematics into key components* highlight two contrasting situations. The first example involved the teacher deconstructing mathematical processes that were routine and familiar, and the second example required the teacher to navigate an unforeseen mathematical situation. These observations resonate with Chick and Stacey’s (2013) assertion that dealing with a mathematics teaching challenge, such as solving exercises from a textbook, is routine because the teacher is already familiar with the solution process. By contrast, other teaching challenges involve addressing mathematical problems for which the solution process is not immediately apparent and therefore calls upon the “complex interplay” of a teacher’s mathematical and pedagogical knowledge (p. 121).

From the researcher’s perspective, the teachers’ connections-related PCK was characterised by their awareness of the relative cognitive demands of procedural aspects of standard textbook problems. In addition, the teachers drew their students’ attention to connections associated with mathematical procedures, but it seemed that they avoided addressing connections that focused on the conceptual underpinnings of mathematical ideas.

5.2.4 Contingency

Rowland et al. (e.g., 2005) use the term contingency to encapsulate the idea that teachers cannot predict everything that will take place in a lesson, and at times events will unfold that will require the teacher to make in-the-moment decisions. From the researcher's perspective, *responding to students' ideas* and *teacher insight* were the key elements of the contingency dimension that were enacted by the teachers.

Responding to students' ideas refers to a teacher's response to a student's unexpected contribution to the lesson (Rowland et al., 2015). For example, in Scenario 1 ("Tom's suggestion") Mr Jones acknowledged, but did not pursue, a student's suggestion of an alternative method for solving a specific optimisation problem, even though the suggestion offered a potentially source of rich discussion with the whole class. As discussed by Schoenfeld (1998), a teacher must weigh up whether he or she deems it appropriate to deviate from the lesson agenda in response to a student's unexpected contribution. Mr Jones' decision not to address Tom's suggestion is elaborated on in Section 5.1.2, from the teacher's perspective.

Another example of *responding to students' ideas* was evident in Scenario 10 ("Introduction to variance") when Mr McLaren, who had initially been reluctant to address why Var $(aX + b)$ is equal to a^2Var (X), was prompted in-the-moment by a student's unexpected query about why the b "disappears". The challenge of drawing upon his own content knowledge on the spot – perhaps within a context he had not previously considered – impacted upon Mr McLaren's capacity to offer a clear and thorough explanation at the time. This kind of improvisation illustrates the complexity of being able to draw upon mathematical knowledge in the moment it is needed.

Mason and Davis (2013) articulate the complexity of such instances by pinpointing the crucial role of the "connective tissue" between "mathematical awareness" (e.g., noticing an absence in understanding from a learner) and in-the-moment pedagogy (e.g., having an appropriate pedagogical action come to mind when needed) (p. 183).

Another element of contingency that involves the teacher in making in-the-moment unplanned teaching decisions, is *teacher insight*. While *responding to students' ideas* is triggered by a student's unexpected contribution to the lesson, *teacher insight* is triggered by the teacher's own thinking (Rowland et al., 2015). It was not always possible for the researcher to recognise in-the-moment teaching decisions, particularly in situations when the teachers' actions appeared seamless, but some instances of *teacher insight* were discernible. For example, in Scenario 13 ("The Tattslotto problem"), Mr McLaren's students had not recognised that $\log_e\left(\frac{8145059}{8145060}\right)$ is negative, which therefore had implications for the inequality they were attempting to solve. As a result, Mr McLaren stopped and paused contemplatively at the board for a few seconds before sketching a general logarithmic graph to illuminate the domain for which the log is negative. Mr McLaren's post-lesson interview responses offered some insight into this instructional choice, as discussed in Section 5.1.2.

The elements within the contingency dimension enabled the researcher to home in on the teachers' in-the-moment pedagogical choices of action. While it was intriguing to observe this kind of PCK unfold during the lessons, additional insights into the motivation and triggers behind these choices and actions were gleaned from the teachers' perspectives as discussed in Section 5.1.2.

Having focused on the classroom enactment of the teachers' PCK as observed by the researcher, the next section examines the teachers' instructional decisions and actions from their own perspective.

5.3 Research Question 2: The PCK behind Instructional Decisions Attributed by the Teachers

The second research question was addressed by analysing the post-lesson teacher interview data. The teacher interviews supplemented and enhanced the researcher's observations because they provided the teachers with the opportunity to elaborate on the reasons for their observed actions and offer their perspective on aspects of the lessons. This allowed teachers to reveal more of the PCK that underpinned their pedagogical choices and classroom actions.

5.3.1 Foundation

The teacher interviews provided supplementary insights into elements of PCK from the foundation dimension that were observed by the researcher including: *knowledge of assessment*, *knowledge of curriculum, use of mathematical terminology*, and *concentration on procedures.* In addition, two other elements from foundation, *theoretical underpinning of the pedagogy* or *knowledge of students' errors*, were brought to light by the teachers during their interview, both of which were not necessarily evident through observation alone.

Knowledge of assessment was identified by the teachers as one of the key factors that influenced their instructional decisions, including their choice and use of specific examples. For instance, in Scenarios 1 and 6 ("Tom's suggestion" and "Using

CAS for integral calculus" respectively), Mr Jones justified his choice of examples based on their suitability as potential examination-type questions. In Scenario 1 he attributed his focus on a specific method of solution to the need to remain "faithful" to the kinds of questions that come up in the examination and the typical ways in which they should be solved. Similarly, in Scenario 7, Mr Jones selected an example that he believed was well-suited to the technology-enabled section of the examination. He explained that his choice of example had been "exam driven a bit", given that students would be expected to use their calculators to find "more complex areas". Although these lesson episodes could have been discussed within the context of *knowledge of examples* (from the transformation dimension), *knowledge of assessment* was particularly relevant given its impact on the teacher's choice of examples.

Evidence from the teachers' interview responses indicate that other elements of foundation knowledge were also intertwined with their *knowledge of assessment* including *knowledge of curriculum* and *use of mathematical terminology*. For instance, in Scenario 4 ("Anti-differentiation with Mr Taylor"), the teacher explained that his decision to emphasise the precise use of integral notation was based on information provided in the annual examiners' reports, that students tend to use the operators $\int$ and dx incorrectly. In this sense, Mr Taylor's *use of mathematical terminology* was linked to his knowledge of the Mathematics Methods course context and therefore to his *knowledge of curriculum* and *knowledge of assessment.*

Further insights into *knowledge of curriculum* and its interactions with other elements of PCK were also offered from the teachers' perspective. For example, *knowledge of curriculum* and *concentration of procedures* were evident in Scenario 8 ("Trig pops up everywhere") when Mr Jones defended his focus on procedures because he perceived the calculus component of the course as mainly focused on the application of skills involving differentiation and integration.

Another key element of PCK from the *foundation* dimension that influenced the teachers' instructional choices, was their *knowledge of student errors*. In the topic of calculus, the teachers' interview responses indicated that they were very aware of the types of procedural errors that students commonly make during algebraic manipulation. In Scenario 9 ("e^2 is just a number") Mr McLaren highlighted the tendency for students to misinterpret the constant e^2 as a variable within the context of anti-differentiation. He also reflected on the source of his knowledge of student errors, by suggesting that the knowledge had developed through the experience of teaching the topic multiple times, and "seeing" the same errors being made by students over time. Similarly, in Scenario 4 ("Anti-differentiation with Mr Taylor), the teacher described the predictable mistakes that students make when learning skills of anti-differentiation, including confusing the steps involved in the processes of differentiation and anti-differentiation. Like Mr McLaren, Mr Taylor attributed his knowledge of student errors to the wisdom of practice, which afforded him the ability to "see in advance what kinds of errors they [the students] are going to make."

Within the domain of probability, the teachers highlighted students' tendencies to have difficulty determining whether a problem requires the application of the binomial or hypergeometric probability distributions. For instance, in Scenario 14 ("The hospital problem"), Mr Jones explained that students often confuse the two distributions when there are subtle differences in the wording of the problems, including interpreting a population from a sample, or the number of successes in a population.

The teachers' post-lesson interview responses suggest they were acutely aware of the errors their students were likely to make in relation to aspects of the Mathematics Methods course content, especially within the topic of calculus. The teachers drew upon their previous experiences with teaching the course when discussing the typical errors that their students make, rather than focusing on theoretically-based sources (e.g., mathematics education research). The source of the teachers' knowledge is noteworthy because Rowland and his colleagues (2005) imply that teachers' foundation knowledge, including *knowledge of students' errors*, is rooted in well-established findings from mathematics education research into effective teaching practices, whereas it seems to arise here from their own experience.

The teachers' perspective of the most effective ways to address students' errors also offered insights into their mathematics-related beliefs. As such, *theoretical underpinning of the pedagogy* provided a useful element with which to classify evidence of the teachers' beliefs about how the Mathematics Methods course content is best learnt and taught. For example, in Scenario 5 ("Anti-differentiation with Mr Taylor") the teacher articulated his aim to engage his students more actively in their own mathematics learning based on his perception of the success they experience by "doing a lot of questioning, talking, and bringing things out." In other words, Mr Taylor suggested that students are more likely to make meaning of mathematical skills and concepts if they grapple with and face potential difficulties by themselves rather than being shown or told. This view of learning and teaching influenced the way in which Mr Taylor introduced ideas relating to the process of anti-differentiation, including his approach to encouraging the students to come to realise "what was different" about $\int 5x^{-1}dx$ in terms of the "initial rules about adding one to the index and then dividing by the new index". Similar beliefs were expressed by Mr McLaren in Scenario 9 ("e^2 is just a number") when he discussed the value of allowing students to recognise and confront their own mathematical errors because they are more likely to take them on board.

Interesting tensions were raised by Mr McLaren in Scenario 7 ("The area from 1 to e) that related to his mathematics-related beliefs. On the one hand he expressed the view that it is "good" for students to see some of the "wonder" of mathematics by exploring interesting ideas and connections, and on the other hand he implied that this was not always possible, given the content-dense nature of the Mathematics Methods course. There was a sense in which Mr McLaren perceived the "interesting snippets" (such as the idea that the under the curve between 1 and e of $y = \frac{1}{x}$ is exactly 1 square unit) as separate from the regular course content and held the view that it is not always possible to attend to both.

The idea that "you can't cover it all" was also expressed in Scenario 4 ("Anti-differentiation with Mr Taylor"). While Mr Taylor acknowledged the value of addressing the conceptual link between integration and the area of regions enclosed by functions, he chose not to, on the grounds that the time available to cover the required course content was already scarce. A similar viewpoint was expressed by Mr Jones in Scenario 14 ("The hospital problem") when he explained that there was limited time available in the course to "stop and smell the roses" by exploring mathematical ideas in more depth. In addition, Mr Jones defended his emphasis on the practical application of discrete probability distributions by asserting that an important component of students' "understanding" is their capacity to choose appropriate tools to solve real world problems and to reach "an end product". In this instance, Mr Jones conveyed the idea that mathematics is a collection of tools used to achieve an external end, an interpretation that appears to align with an instrumentalist view (Ernest, 1983) of mathematics.

In other words, Mr Jones, Mr Taylor, and Mr McLaren implied that constraints around meeting the course requirements limited the opportunity to focus on some of the deeper underlying ideas of mathematics, reflecting their knowledge of curriculum as well as their *theoretical underpinning of the pedagogy*. The examples featuring the teachers' *theoretical underpinning of the pedagogy* are supported by literature (e.g., Hashweh, 2005; Magnusson et al., 1999) that acknowledges the central role that teachers' conceptions of the purpose of specific content, and the ways in which it should be taught and learnt, play out in their teaching.

5.3.2 Transformation

The main elements of PCK that the teachers discussed during their post-lesson interviews, from the transformation dimension, included *knowledge of examples* and *use of instructional resources*. The teachers' responses corroborated, and in some cases elaborated upon, the researcher's perspective on these elements of PCK.

Knowledge of examples from the teachers' perspective was evident in their justification for choosing specific examples for whole class demonstration or for the students' individual skills practice. Some post-lesson interview responses indicated that the teachers chose examples based on their awareness of what the students were likely to find procedurally challenging. For example, in Scenario 5 ("It's not a perfect square") Mr Jones explained that he had deliberately selected an example because it involved a "difference of two squares" situation that would not be recognised immediately by many students. Similarly, in Scenario 4 ("Anti-differentiation with

Mr Taylor) the teacher explained that he had sequenced a set of examples in order of their procedural difficulty, a teaching action that was also noticed by the researcher during her lesson observation. In addition, Mr Taylor elaborated on his choice and sequencing of examples by making the point that seemingly small changes made to an example can profoundly change its meaning and complexity. In this instance, Mr Taylor's *knowledge of examples* was tied in with his awareness of the cognitive demands of the tasks and therefore linked to *anticipation of complexit*y from the connection dimension.

The teachers' post-lesson interviews also provided useful additional perspective on their *use of instructional resources.* On some occasions, the teachers elaborated on the ways in which they modified and supplemented materials from the textbook. For example, in Scenario 11 ("It's Pascal's Triangle"), Mr McLaren discussed an activity from the prescribed text book designed to illuminate the connection between the binomial probability distribution and the binomial theorem. The activity included a table showing the pattern of outcomes relating to the probability of a three appearing uppermost 0, 1, 2, 3, and 4 times when a fair die is rolled. Mr McLaren explained that he had presented the table in parts to enable the students to "see" the pattern of probabilities "unfold" rather than presenting them with the entire table upfront, as it appeared in the textbook. In this case, Mr McLaren's *use of instructional resources*, related to the way in which he adapted the textbook material in a pedagogically useful way. It is worth noting that the researcher considered this instance to align more appropriately to *use of instructional resources* rather than *adherence to textbooks* from the foundation category. Mr McLaren modified the delivery of the textbook material in a way that both the researcher and the teacher believe was more meaningful to the students. As such, the instance was more about transformation than the teacher's foundation knowledge alone.

The post-lesson interviews also offered additional perspective on the teachers' use of CAS as an instructional resource. For example, in Scenario 12 ("Using CAS to explore skewness") Mr McLaren elaborated on his attempt to use CAS to investigate skewness of the binomial probability distribution. He remarked that the lesson had not gone as well as he had hoped and attributed the limitations of the lesson largely to the absence of the CAS emulator. The emulator was a tool often used by Mr McLaren to project an interactive display of the calculator screen to provide a visual demonstration of the keystrokes required to solve specific problems. From the researcher's perspective, however, other issues relating to the effective pedagogical use of technology were also evident. As highlighted in Section 5.1.1, a sophisticated kind of PCK is required for a teacher to notice, and act upon, opportunities offered by technology to draw out key ideas.

5.3.3 Connection

Anticipation of complexity and *mathematical structure and connections* were the key elements of PCK from the connection dimension that were discussed by the teachers during their post-lesson interviews. In relation to *anticipation of complexity*, the teachers alluded to some of the tensions relating to decisions about if and how to address complex mathematical ideas.

Anticipation of complexity was evident in Scenario 5 ("It's not a perfect square") when Mr Jones articulated the tensions associated with knowing how quickly to intervene when students are presented with mathematically challenging situations. Ultimately, Mr Jones justified his tendency to "troubleshoot" for the students by saying he did not want to “hold them up” given that other course content also needed to be addressed within the available lesson time.

Other tensions were raised by the teachers that were specific to teaching the topic of probability. For example, during his interview in Scenario 10 (“Introduction to variance”), Mr McLaren remarked that the concept of variance is difficult to teach and to learn. While he attributed this complexity, at least in part, to making sense of the formula for calculating variance, he was also aware that the course did not require the students to unpack the formula. He also discussed the complexity of teaching and learning probability more generally, by pointing out that even though probability seems the “shortest” unit in the course and in a sense “the most straight-forward”, some concepts, particularly those related to measures of spread, were surprisingly challenging.

Interestingly, there appeared to be some differences in the ways in which the teachers discussed the complexity of probability compared to that of calculus. The teachers’ *anticipation of complexity* in the topic of probability tended to focus on challenges associated with making sense of ideas (e.g., conceptualising the formula for variance as a measure of spread). By contrast, their *anticipation of complexity* in relation to calculus focused primarily on procedural aspects of differentiation and integration that students often found difficult. This distinction, however, was not entirely clear-cut given that Mr Jones also identified the Fundamental Theorem of Calculus as “not an easy thing to understand”.

The other key element of the connection dimension that the teachers discussed during their post-lesson interviews was *mathematical structure and connections*. For example, in Scenario 4 (“Anti-differentiation with Mr Taylor”) the teacher discussed his focus on highlighting the relationship between the processes of differentiation and anti-differentiation, rather than treating the latter as "something new". Similarly, in Scenario 6 ("It's not a perfect square"), Mr Jones initiated a discussion about the way in which he had assisted his students to recognise that $8 - x^2$ can be expressed as a "difference of two squares" even though 8 is not a square number.

The teachers' justification for an absence of *mathematical structure and connections* in some of their instructional choices and actions supported and clarified the researcher's observations. In Scenario 8 ("Trig pops up everywhere"), Mr Jones defended his decision not to address the Fundamental Theorem of Calculus in his introduction to calculating the area enclosed by functions. He explained that the course required students to use integral calculus skills in practical contexts, rather than focus on underpinning concepts. The absence of *mathematical structure and connections*, in this instance, was underpinned by the view that students' abilities to apply integral calculus successfully within the Mathematics Methods course did not require them to understand the conceptual and theoretical links between integral calculus and area.

Additional insights into the researcher's observations of an absence of *mathematical structure and connections* in Scenario 14 ("The hospital problem") were offered from Mr Jones' perspective. During the lesson, Mr Jones had emphasised the selection of the correct probability distribution and the individual steps required to solve the "hospital problem". He did not, however, take the opportunity to address the key probability concepts inherent in the problem (e.g., sampling, sample size, and variation). By his own admission, Mr Jones claimed that he was not "across it enough" himself to have addressed why the small hospital was more likely to have greater variation in the proportion of baby boys born than the large hospital. In this case, Mr Jones attributed this oversight, at least in part, to limitations in his own content knowledge.

5.3.4 Contingency

Evidence of *responding to students' ideas* and *teacher insight* were provided in the post-lesson teacher interviews and assisted to refine the observations made by the researcher.

Insights into *responding to students' ideas* were gleaned in Scenario 1 ("Tom's suggestion") when Mr Jones elaborated on why he had not pursued Tom's suggestion to use geometry rather than calculus to solve "the distance problem", a typical optimisation problem. While Mr Jones credited Tom for making a connection to previously learnt content, he firmly pointed out that questions such as the "distance problem" would appear on the calculus section of the examination, and therefore must be solved using calculus. In this sense, the teacher's decision not to pursue Tom's suggestion was influenced by his *knowledge of assessment*.

Mr McLaren's perspective on *responding to students' ideas* in Scenario 10 ("Introduction to variance") clarified the researcher's observations which were discussed in Section 5.1.1. During his interview, Mr McLaren reflected on the way in which he had responded to Grace's unexpected question about why the "*b* disappears" in the relationship $\text{Var}(aX + b) = a^2\text{Var}(X)$. He was able to construct, during the interview, a more thorough and complete explanation as highlighted in the Scenario. Mr McLaren's interview responses suggest that he had reflected-on-action by considering the mathematical connections that did not appear to come to mind in the moment of teaching.

5.4 Research Question 3: Student Identification of PCK

An important feature of this study's design was the generation of data showing evidence of the teachers' PCK from multiple perspectives, including that of the students. The third research question sought to investigate if students were able to recognise aspects of their teachers' PCK, and, if so, the extent to which they found those aspects useful for their learning. The student perspective was explored by analysing the focus-group interview responses and the short-written reflections from the students. Findings suggest that the students identified elements of PCK that were predominantly from the transformation and connection dimensions, which is not surprising given that these dimensions, as highlighted by Rowland et al. (2005), are concerned with the ways in which teachers make mathematics content accessible.

5.4.1 Transformation

Teacher demonstration and *knowledge of representations* were the main elements of PCK, from the transformation dimension, that were addressed from the students' point of view. The focus-group interview responses and short reflections indicated that the students appreciated detailed, step-by-step explanations of the solutions to standard textbook problems. For example, in Scenario 1 ("Tom's suggestion") several students remarked that Mr Jones' whiteboard demonstration of the algebraic processes involved in solving the "distance problem" was especially useful. Similarly, in Scenario 14 ("The hospital problem") the students highlighted the value of being shown how to identify the correct probability distribution to apply in a given problem. One student explained that Mr Jones' demonstrations enabled him to solve the problems more quickly because they helped him to determine the correct formula to use. Other research findings, most notably those related to the Learner Perspective Study (e.g., Huang & Barlow, 2013) which involved studies within the international context, found that students placed high importance on clear teacher demonstrations and explanations of mathematical procedures.

In this study, the students particularly noticed and valued their teachers' worked solutions to problems that were perceived to be procedurally challenging. For instance, Scenarios 3 and 4 illustrate the ways in which Mr Jones and Mr McLaren, respectively, helped their students to solve "the particle problem". Although the two teachers used different methods to simplify a procedurally difficult expression, students from both classes commented that their teacher's worked solution was useful. In Scenario 4, Carl seemed intrigued when Mr McLaren simplified the expression, $\frac{[3(3t^2+4)^{\frac{1}{2}} - 3t\left(3t^2+4\right)^{-\frac{1}{2}} \text{x}\, 3t]}{((3t^2+4)^{\frac{1}{2}})^2}$ by multiplying it by $\frac{(3t^2+4)^{\frac{1}{2}}}{(3t^2+4)^{\frac{1}{2}}}$. In his short-reflection Carl expressed an interest in Mr McLaren's flexible and efficient use of procedures by alluding to the value of being able to "think outside the box when simplifying something". Carl's perspective draws attention to the idea that Mr McLaren modelled the kind of knowledge that Star (2005) refers to as deep procedural knowledge, whereby mathematical procedures are used in versatile and strategic ways rather than being limited to well-practiced or rote learnt techniques.

While Mr Jones' method of simplifying the difficult expression (for the second derivative) was arguably less efficient, his students highlighted that his detailed step-by-step explanation, especially in relation to dealing with negative fractional indices, was particularly useful for their learning. Interestingly, Mr Jones' worked solution to "the particle problem" dominated the focus of the students' interview and written responses in relation to the lesson featured in Scenario 3.

Useful insights into *teacher demonstration* from the student voice were also evident in Scenario 4 ("Anti-differentiation with Mr Taylor"). The students commented favourably on the way in which their teacher had introduced the process of anti-differentiation of expressions of the form ax^n ($n \neq -1$) by accompanying the steps with written annotations (e.g., "increase the index by one" and "divide by the new index"). Although this approach was not seen as particularly significant by the researcher at the time of observation, several students emphasised that the annotations enabled them to "see what is happening" in the problem. The students elaborated by explaining that they often had trouble interpreting the relationship between the steps of demonstrated worked solutions, especially upon later review.

The students valued their teacher's use of graphs and diagrams during whole class demonstrations. As such, *knowledge of representations* was an aspect of PCK that the students particularly noticed and discussed. In Scenario 14 ("The Tattslotto problem") several students commented that Mr McLaren's graph of $y = log_a x$ was useful because it enabled them to realise that $\log_e\left(\frac{8145059}{8145060}\right)$ has a negative value.

Other evidence of *knowledge of representations* from the student perspective, related to the teachers' routine use of representations including diagrams to model mathematical situations. For example, Keira and Alan discussed the clarity with which Mr Jones' diagrams represented the context of specific problems presented in Scenario 1 ("Tom's suggestion"). From the researcher's perspective the use of diagrams in teacher demonstrations seemed unremarkable at the time of observation but was especially noticed and valued by the students.

Other instances of *knowledge of representations* were identified as noteworthy by the researcher but received limited attention from the students in their focus-groups and written reflections. For example, in Scenario 10 (“Introduction to variance”) the students offered little information about the extent to which they found Mr McLaren’s pair of graphs useful for their learning of the concept of variance. While one student commented that the graphs provided helpful visual representations of the distributions, the other students highlighted learning how to use the formula for variance as the most useful aspect of the lesson. That is, the students tended to focus on those aspects of the lesson that related to solving problems and completing exercises from the textbook, rather than on developing mathematical ideas.

5.4.2 Connection

Mathematical structure and connections was the most significant element of PCK, from the connection dimension, that the students discussed in their focus group interviews and short reflections. The students appreciated explanations that highlighted connections among procedures, especially those related to algebraic manipulation. In Scenario 4 (“Anti-differentiation with Mr Taylor”) the students expressed appreciation for the way in which their teacher had introduced the process of anti-differentiation by illuminating its connections with differentiation, an approach that was also noticed by the researcher and discussed by the teacher in his interview. Similarly, in Scenario 5 (“It’s not a perfect square”) the students valued the way in which Mr Jones had assisted them to recognise that the difference of two squares method of factorisation generalises beyond square numbers. This teaching and learning episode was also observed by the researcher and discussed by Mr Jones as highlighted in Sections 5.1.1 and 5.1.2.

Insights into situations that involved an absence of *mathematical structure and connections* were also provided from the student perspective. A case in point was

Scenario 10 ("Introduction to variance") when Kale highlighted that Mr McLaren's decision to omit the proof of $\mathrm{E}(X)^2$ - $[\mathrm{E}(X)]^2 = \sum(x - \mu)^2\mathrm{Pr}(X{=}x)$ was particularly helpful because it avoided causing confusion for the class, particularly given that they already had new and procedurally challenging content to contend with.

The students offered useful insights into *classroom techniques* in terms of how the teachers helped them to make connections and confront difficult mathematical processes. During their focus-group in Scenario 9 ("e^2 is just a number") the students valued the way Mr McLaren allowed them to "get a bit stumped" before providing subtle and timely prompts to help them to overcome mathematical "hurdles". The students' interpretation of Mr McLaren's teaching technique corroborated the researcher's observations and the teacher's commentary on his instructional choices of action.

This alignment between the teacher's justification for using a particular teaching approach and the students' perception of the value of the approach, relates to Huang and Barlow's (2013) research findings. These authors found that the students in their study particularly noticed those lesson events that were intentionally designed by their teacher to help them to overcome difficulties or to highlight key aspects of the content.

5.5 Summary of the Findings

The three research questions informed the design and implementation of a fine-grained and systematic exploration into PCK at the senior secondary mathematics level, from multiple perspectives. This section provides a summary of the findings of the study by drawing together the key ideas that were discussed, in relation to the three research questions, in the previous sections.

Multiple and intertwined aspects of PCK were observed by the researcher and discussed by the teachers and their students. The teachers displayed strong knowledge of the content of the Mathematics Methods course. *Overt display of subject matter knowledge*, however, offered a useful filter through which to explore the enactment of aspects of the teachers' content knowledge more closely, including instances where the teachers drew upon their knowledge in particularly explicit ways. In some cases, there were differences in the fluency of the teacher's *overt display of subject matter knowledge* depending on the mathematical situation. For instance, Mr McLaren seamlessly drew upon his mathematical knowledge to sketch the log graph in Scenario14 ("The Tattslotto problems") but seemed less familiar (although clearly aware of the general principles involved) with unpacking the relationship $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$. This distinction is perhaps not surprising because Mr McLaren was not usually called upon to address the derivation of variance-related formulae in this way.

A key way in which the teachers drew upon their own mathematical content knowledge was in the selection of instructional examples to illuminate specific mathematical ideas and the ways in which they used examples in the prescribed textbook in strategic ways. The teachers' abundant use of examples, both for instructional purposes and for students' skills practice, was a prominent aspect of their PCK. The researcher's observations were supplemented by the teachers' perspective on key factors that had influenced their choice and use of examples, including their awareness of the requirements of the external examination.

The teachers' demonstrations were characterised by detailed explanations of the processes involved in solving typical text-book and examination-type problems, a practice that was noticed and appreciated by the students. In general, the students valued step-by-step explanations of how to do specific problems, including those involving the use of the CAS calculator. For example, the students commented on the usefulness of demonstrations that showed them how to use the calculator as a functional tool to solve technology-enabled problems, whereas they rarely remarked on their teacher's pedagogical use of technology to develop mathematical ideas (e.g., Mr McLaren's graphs used to convey the idea of variance).

Particularly evident from the teachers' perspective was the teachers' acute awareness of the common errors that their students were likely to make when learning new skills and solving problems. The teachers made a point of attributing their knowledge of student errors to the wisdom of practice, including their familiarity with specific exercises from the prescribed textbook.

In some instances, the teachers' *knowledge of student errors* was related to their *anticipation of complexity* in relation to specific tasks. The ways in which the teachers handled this complexity seemed to align with their expressed beliefs about how mathematics is best taught and learnt. For example, the researcher noticed that Mr Taylor and Mr McLaren regularly encouraged their students to confront and address their own mathematical errors. Correspondingly, the teachers justified their approach by explaining that students are likely to learn better if they work through challenges for themselves, rather than being told from the outset. Several students, particularly those from Mr McLaren's class, supported this view and were able to articulate their teacher's intention to help them to "discover it for themselves".
By contrast, Mr Jones was reluctant to let his students wrestle with difficult mathematical skills and ideas based on his perception of the Mathematics Methods course context, including meeting the requirements of a content-dense course within a limited timeframe.

In general, the teachers managed the complexity of mathematical tasks by either reducing the cognitive demand of the tasks, or by supporting the students to grapple with the complexity themselves. Several scholars (e.g., Anthony, 1996; Henningsen & Stein, 1997) suggest that when teachers routinely and systematically reduce the cognitive demand of mathematical tasks, the depth of autonomous thinking required by the students is, in turn, reduced. In addition, research into the quality of student engagement with mathematical tasks (e.g., Henningsen & Stein, 1997) suggests that teachers should actively support their students' mathematical activity without reducing the cognitive demands of the task.

The teachers and the students tended to privilege procedural mastery, especially in situations that were perceived as challenging. A key aspect of the teachers' PCK was their emphasis on drawing connections among algebraic procedures. This practice was observed by the researcher and highlighted as important by both the teachers and their students. In other situations, the teachers regularly chose not to focus on mathematical connections related to concepts underpinning the mathematics content. For the most part, the teachers attributed these decisions to key factors related to the Mathematics Methods course, including its focus on the application of the mathematics rather than on underlying concepts, and the pressure of covering the required course content within a limited timeframe. A case in point was Mr McLaren's decision to omit of the proof of the variance formula. These instructional decisions suggest that knowledge of the procedural demands of "typical" examination and text-book problems are part of a teacher's PCK and affect priorities about what is taught and attended to by students. In some instances, the teachers expressed a lack of confidence in their own knowledge of the concepts underpinning specific mathematical procedures, which has implications for Mason's (2009) notion of a richly conceived PCK that encourages teachers to "be mathematical with and in front of their learners" (p. 307).

The idea of being mathematical "with and in front of the learner" was relevant to the ways in which the teachers responded to unexpected questions or ideas from their students. In some instances, the teachers opted to acknowledge, but not pursue, unexpected contributions that were not directly related to the course content. The rare occasions when a teacher *did* attempt to address a student's unexpected question served to highlight the complexity involved in drawing upon sophisticated mathematical knowledge in the moment of teaching and transforming it in pedagogically useful ways for the student. This finding aligns with Potari and her colleagues' (2007) claim that the role of coordinating content and pedagogy becomes more complex as the mathematics content becomes more advanced. In addition, Mason and Davis (2013) claim that a teacher's own mathematical thinking is one of the most important factors in their capacity to engage productively with their students in moment-by-moment classroom interactions.

The findings of the study indicate that multiple and interconnected aspects of PCK were enacted by the teachers to transform the course content in ways that made it accessible for their students. The teachers' own commentary on their instructional decisions focused on their perception of the constraints of the context of the Mathematics Methods syllabus, particularly in relation to the high-stakes external examination. The teachers' in-the-moment pedagogical choices of action were also influenced by their perception of course constraints, and in some cases by limitations in their own knowledge of mathematical ideas that were beyond the confines of the content of Mathematics Methods.

5.5.1 Summary of the Chapter

This chapter addressed the three research questions separately by drawing upon the results presented in the Scenarios described in Chapter 4. The summary in the previous section was designed to bring together and discuss the key aspects of PCK from each of the three perspectives. The next chapter provides a concluding analysis of the study's design, key findings, contribution, and implications for current and future research.

Chapter 6

Conclusion and Implications

6.1 Overview of the Study

This study was motivated by the researcher's interest in the mathematics-related knowledge held and used by experienced teachers to convey abstract mathematical ideas that are characteristic of senior secondary mathematics courses. Based on the assumption that this type of knowledge transcends the teachers' knowledge of the mathematics content itself, the most useful construct with which to examine the enactment of senior secondary mathematics teaching and learning was PCK.

Views on PCK (e.g., Hashweh, 2005; Magnusson, et al., 1999; Mason & Davis, 2013; Shulman, 2015) incorporate multiple aspects of teachers' work, from their initial planning to their moment-by-moment pedagogical choices and actions during teaching. In addition, these views on PCK acknowledge the ways in which external factors, such as broader classroom and societal contexts, impact teachers' instructional decisions.

Multiple sources of evidence of PCK were generated from the perspectives of the researcher, the participating teachers, and their students. These sources of evidence included lesson observations and video footage of the lessons, interviews with participating teachers, and focus-groups and short written reflections from students. Given that teaching and learning are multi-dimensional social phenomena, the generation of data from different perspectives was crucial for the purposes of examining evidence of PCK in a multifaceted way.

Data showing evidence of the teachers' PCK from the three perspectives (researcher, teacher, and student) were coded and analysed using a combination of inductive and deductive approaches informed by existing theoretical frameworks, the Knowledge Quartet (Rowland et al., 2005) and the Chick et al. PCK framework (e.g., Chick et al., 2006). The data were qualitatively described and presented in the form of scenarios. Each scenario consisted of a detailed snapshot of a lesson episode described from the researcher's perspective, and responses from the teacher and his students corresponding to the lesson episode. The 14 scenarios were selected on the basis that they provided evidence of the PCK typically demonstrated by the teachers in this study, and/or because they presented interesting tensions relating to PCK at the senior secondary mathematics level.

While the researcher's influence on the selection and construction of the scenarios was inevitable, steps were taken to provide evidence of aspects of PCK that were typically observed during the study. There were no significant incidents or apparent implications of any aspects of teacher knowledge that could not be categorised by the PCK elements from either the KQ or the Chick et al. PCK framework. The structure of the Scenarios enabled the researcher to present the complexity of the PCK evident in the observed lessons, by providing thick descriptions of the teaching and learning observed in the lessons, accompanied by specific insights from the viewpoints of both teacher and students.

6.2 Overview of the Findings

This exploratory study described, analysed, and discussed the enactment of PCK in three senior secondary mathematics classrooms. The investigation was guided by the three research questions (listed below) to generate and examine rich evidence of PCK from multiple perspectives.

1. What aspects of mathematical pedagogical content knowledge are evident in the interactions between teachers and their students during the teaching and learning of senior secondary mathematics content?
2. What aspects of mathematical pedagogical content knowledge do teachers discuss and attribute their instructional decisions to when analysing their interactions with students during the teaching and learning of senior secondary mathematics content?
3. What aspects of mathematical pedagogical content knowledge are identified by students as having an impact on their learning of senior secondary mathematics content?

It is important to highlight that the researcher's perspective was implicated in all three research questions. The first research question is clearly linked to the researcher's perspective because it is concerned with observed classroom practice. While the other two research questions are primarily concerned with the teacher and student voice respectively, they are not strictly free of the researcher's perspective. The researcher's viewpoint came into play in her interpretation of the teachers' and/or students' comments.

The summary of key findings presented in this section draws upon the insights gleaned from all three perspectives (the researcher, the teachers, and the students) to present a holistic interpretation of the senior secondary mathematics teachers' PCK. For example, at times the multiple perspectives corroborated each other, where specific lesson events were noticed by the researcher and discussed by the teacher and the students. In addition, the teachers' perspectives often supplemented and informed the researcher's observations. Of interest, also, were differences in the researcher's and the students' perspectives on significant aspects of a lesson. For example, in some cases (e.g., Scenarios 7 "The area from 1 to *e*", 10 "Introduction to variance", and 12 "Using CAS to explore skewness) the researcher observed the teacher's focus on the development and exploration of mathematical ideas, whereas the students did not comment on the incident in their focus-groups or written responses.

In general, however, there was alignment of responses among students and teacher. For example, Mr Jones and his students tended to privilege procedural mastery, particularly in situations involving challenging algebraic manipulation, whereas Mr McLaren and his students highlighted the value of having the opportunity to "discover" mathematical ideas and confront difficulties instead of being told or shown upfront. Interestingly the exam-focused and time-poor narrative that was common in the teachers' interview responses, was not directly reflected in the students' responses, although they did, in general, favour aspects of their teacher's PCK that helped them to solve standard problems.

Findings suggested that the teachers possessed a repertoire of PCK (e.g., *knowledge of examples*, *knowledge of instructional resources*, *knowledge of assessment*) that was geared towards meeting the requirements of the Mathematics Methods course, particularly in relation to the high-stakes external examination. From the researcher's perspective, the teachers provided detailed, step-by-step demonstrations of mathematical procedures, a practice that was especially noticed and valued by the students. The teachers made a point of attributing their knowledge of student errors, and their awareness of the cognitive demands of different tasks, to a wisdom of practice as experienced teachers of Mathematics Methods. In some instances, the students particularly recognised and commented on the strategies their teacher used to address the common errors they were likely to make and difficulties they would experience with some content.

From the researcher's perspective, the teachers regularly drew attention to connections between mathematical processes but tended not to address connections that focused on underlying concepts or foundational ideas underpinning the mathematics. The teachers often attributed these instructional decisions to the perceived constraints of the Mathematics Methods course. These constraints included the course's emphasis on applications rather than underlying concepts of mathematics, completing the course content within the allocated timeframe, and the expectations of the high-stakes external examination.

In some circumstances, the teachers expressed a lack of confidence in their own capacity to unpack and draw connections among ideas that underpinned specific content which was not directly a focus of the course. One exception was Mr McLaren's attempts, on occasions, to explore and develop ideas related to the topic of probability. The researcher viewed these aspects of Mr McLaren's lessons as pedagogically significant, but they were not discussed by the students in their interviews and written responses.

In general, the teachers tended to privilege performance of procedures to solve routine problems and were reluctant to deviate far from these standard approaches.

This tendency to adhere to standard procedures for solving conventional problems raises questions about the extent to which the teachers have opportunities to develop the kind of PCK that Mason associates with "being mathematical with and in front of learners" (2008, p. 20). The idea of being mathematical with and in front of the learner is concerned with the teacher's sensitivity to and awareness of opportunities to initiate action or mediate classroom activity in ways that enable students to become aware of relevant and important aspects of the mathematics at hand (2008). This focus on sensitivity and awareness implies that the teacher must be intensely present in moment-by-moment classroom interactions to integrate content and pedagogy in rich and complex ways. While not an explicit focus of this study, it is interesting to consider the teachers' perceptions of "being mathematical" and the extent to which their mathematics-related beliefs influenced these perceptions and ultimately their instructional choices. For example, Mr Jones' expressed pragmatic views of teaching and learning that focused on applying mathematics to solve practical problems (e.g., Scenario 13 "The hospital problem"), whereas Mr McLaren, in Scenario 7 ("Area under curve from 1 to *k*"), placed emphasis on wanting to show the students some of the "wonder" of mathematics.

6.3 Limitations of the Study

This section addresses key limitations of the investigation, some of which are inherent in the research design, and others are related to circumstances that arose during the implementation of the study. Underpinned by an interpretivist paradigm, all aspects of the research process were inevitably filtered through the researcher's interpretation. As highlighted by Mason (2002), qualitative researchers can never be neutral or detached from the data they are generating, analysing, and presenting. It was therefore important to scrutinise the trustworthines of the study at all stages of the investigation as discussed in Section 3.10.

During the post-lesson interviews and focus groups, the researcher often noticed and pursued interesting points related to PCK that were raised by the participants. On some occasions, however, the researcher realised, after the interview was over and it was too late to revisit, that other questions could have been asked to probe for deeper insights or clarification. This "l'esprit d'escalier" suggests that what comes to mind in-the-moment for a researcher conducting an interview can be as unpredictable as what comes to mind in-the-moment for a teacher during an episode of contingency.

Although this study was limited by the small number of teachers, the inclusion of more participants would have rendered the study unmanageable, given the high volume of qualitative data that would be generated and analysed by one researcher. It would, however, also have been interesting to observe the enactment of PCK by a teacher who used different or more innovative approaches to teaching and learning the content of Mathematics Methods. The three teachers in this study used similar traditional approaches to teaching the course, including whole-class demonstrations of standard textbook problems, followed by individual seatwork.

Given the high stakes nature of Mathematics Methods as an externally assessed course, the researcher needed to be mindful that the students' participation impacted as little as possible on their studies. In addition, limited data were provided by the students in Mr Taylor's class because most of their lessons were scheduled in the afternoon leading up to the end of the school day when students needed to catch buses or meet co-curricular commitments. This situation meant that there was little time and opportunity for the researcher to interview and obtain written reflections from Mr Taylor's students, which reduced the student perspective on his lessons.

Given that the analysis of the data generated was localised to the 18 lessons comprising this collective case study, it was not possible to provide a picture of the entire school year, or every aspect of the teachers' PCK. As such this study did not attempt to illustrate every occurrence of PCK in an exhaustive way, or to micro-analyse instances where multiple aspects of teacher knowledge were at play, or to examine other topics in the curriculum. Nevertheless, the complexity and overlap of different aspects of PCK depicted in the Scenarios suggest that such analysis is necessarily complicated and reflects the realities of teaching, and that this study was able to examine PCK at a suitable depth to reveal this complexity.

6.4 The Use of the Frameworks

A combination of components from the Knowledge Quartet and the Chick et al. frameworks were used to deductively code and analyse evidence of PCK in the multiple sources of data generated. The KQ provided a well-defined structure upon which to explore the enactment of PCK at the senior secondary level, including the knowledge held by the teacher irrespective of its implementation in the classroom (foundation dimension), the knowledge used to transform mathematics content for teaching (transformation and connection dimensions), and the teacher's in-the-moment pedagogical choices of action during teaching (contingency dimension). This structure was useful for examining the range of different elements of PCK evident from each of the three perspectives.

Elements of PCK from all four dimensions of the KQ were observed by the researcher and discussed by the teachers. The richest source of evidence of PCK from the foundation dimension, however, was the teachers' post-lesson interviews. This is not surprising given that the foundation dimension is concerned with knowledge held by the teacher regardless of its enactment and is therefore not necessarily evident through observation alone. Interestingly, the student responses focused on elements of PCK from transformation and connection, which also makes sense given that these dimensions are concerned with the ways in which teachers make content accessible to the learner.

Although the KQ was designed to explore the enactment of preservice primary teachers' mathematics-related knowledge, the framework proved useful in this study for analysing aspects of PCK at the senior secondary mathematics level. For example, *adherence to textbooks* was originally used as a filter through which to examine the extent to which a novice or pre-service primary teachers can make informed decisions about when it is appropriate to adhere to or deviate from the material presented in the textbook (Rowland et al., 2009). This element, however, was also useful in the exploration of ways in which the experienced senior secondary mathematics teachers in this study, who were very familiar with the content of the Mathematics Methods course, adhered to the prescribed textbook. Of note was Mr Taylor's purposeful adherence to the way in which the textbook distinguished *an* antiderivative from *the* antiderivative as featured in Scenario 4.

Similarly, *overt display of subject matter knowledge* offered a useful lens through which to examine the enactment of the teachers' mathematical content knowledge in nuanced ways. This element was particularly useful for homing in on the enactment of content knowledge that transcended the teachers' routine display of content knowledge and part of their everyday work in the classroom. For example, Scenarios 1 ("Tom' suggestion"), 3 ("The particle problem: Mr McLaren"), 10

(“Introduction to variance”), and 13 (“The hospital problem”) illustrate evidence of overt display of subject matter knowledge, including from the perspective of its absence as highlighted in Scenarios 1 and 13.

The KQ comprises a range of components that were highly relevant to this study, especially those related to prototypic aspects of PCK (e.g., Shulman, 1986) such as knowledge of student thinking, anticipating complexity, and transforming content for teaching. This framework alone, however, did not adequately address the full spectrum of PCK enacted by the senior secondary mathematics teachers in this study. As such, several components from the Chick et al. (e.g., 2006) PCK framework provided useful additional codes, particularly in relation to addressing knowledge of assessment, mathematical structure and connections, and general classroom techniques. A more detailed justification for the use of these aspects of the Chick et al. framework, was provided in Section 3.9.2.

Given the sophisticated and multifaceted nature of teacher knowledge, any theoretical models used to explore this knowledge, should themselves be complex and nuanced. Collectively, the elements of the KQ and Chick et al. frameworks offered a fine-grained set of filters through which to identify, describe, and analyse the PCK enacted by the teachers. Multiple elements of the frameworks were needed to code each teaching and learning episode.

This overlap among elements reflects the complexity of mathematics-related knowledge enacted by the teachers. For example, a teaching and learning episode where the teacher introduced an example to his students by explaining its significance to the external examination, would be coded as both *knowledge of examples* and *knowledge of assessment.* Overlap among elements of PCK was also apparent in more complex and intricate ways, and from different perspectives. For instance, in Scenario

14 (“The hospital problem”) Mr Jones’ *concentration on procedures* was associated with his *theoretical underpinning of the pedagogy* and *knowledge of curriculum*, as well as an absence of both *mathematical structure and connections* and *overt display of subject matter knowledge*. That is, from the researcher’s perspective, Mr Jones’ focus on the steps involved in using the binomial probability distribution to solve “the hospital problem” was coded as *concentration on procedures.* In addition, he overlooked the opportunity to address the foundational ideas of sampling and variation that were inherent in the problem, an observation that was identified as an absence of *mathematical structure and connections*. From Mr Jones’ perspective this oversight was linked to a lack of confidence in his own capacity to address these ideas indicating an absence of *overt display of subject matter knowledge*. *Theoretical underpinning of the pedagogy* was evident when he expressed the belief that students’ ability to choose and use the correct tools to solve problems was a key priority of learning mathematics. *Knowledge of curriculum* was also evident when Mr Jones pointed out that the course emphasised the application of skills and formulae, rather than deeper ideas underpinning the mathematics.

This illustrative example, featuring Scenario 14, highlights the necessity to draw upon different elements of the frameworks to unravel the complex nature of PCK. In other words, the frameworks provided a powerful way of illuminating the interactions among multiple elements of PCK. These findings relating to the use of the frameworks support the work of scholars (e.g., Fennema & Franke, 1992; Hashweh, 2005) who emphasise the dynamic nature of PCK and the inherent interplay among categories of teacher knowledge.

6.5 Contributions of the Study

This investigation contributes to the field of research into mathematics-related teacher knowledge, by providing insights into PCK at the senior secondary mathematics level, an area that has been previously under-researched. The study contributes to current research in the following ways: by providing evidence of the appropriateness of using existing teacher-knowledge frameworks at the senior secondary mathematics, by including the student perspective in the analysis of PCK, by highlighting the impact of the course context on the teachers' PCK, and by providing a model for presenting and analysing the enactment of PCK using thick description.

The KQ and the Chick et al. framework have mainly been used in the primary mathematics classroom setting (e.g., Baker & Chick, 2006; Chick et al., 2006; Rowland et al., 2005). As highlighted in Section 6.3, the KQ offered a useful structure and a range of relevant components with which to code and analyse evidence of PCK in the senior secondary mathematics classroom. Additional elements from the Chick et al. framework, were used to code aspects of PCK that were evident in the data, but not explicitly identified in the KQ (e.g., knowledge of assessment).

The Chick et al. framework also acknowledges the role of generic teaching practices (e.g., questioning, wait time) in the generation of PCK. The idea that general pedagogical knowledge engenders PCK when it is enacted in specific content contexts was relevant to this study and is underpinned by the work of Marks (1999). As such, *classroom techniques* from the "pedagogical knowledge in a content context" section of the Chick et al. framework provided a useful filter through which to explore the teachers' PCK in action, especially from the students' point of view.

The inclusion of the student perspective in this exploration into PCK in the senior secondary classroom contributes to existing research into the role of the student voice in examining the teaching and learning of mathematics. Most notably, the Learner Perspective Study (e.g., Anthony et al., 2013; Huang & Barlow, 2013) presents findings from a range of international studies that point to the valuable insights offered by students' perceptions of the nature of their own mathematics classrooms. Admittedly, in this study, the time available to generate data from the student perspective was limited given that Mathematics Methods was a high-stakes externally assessed course and it was important that students' participation in this research impacted their studies as little as possible. Notwithstanding these limitations, findings indicated that the students noticed aspects of their teachers' PCK and were able to articulate, to some extent, those aspects that were particularly useful for their learning.

In this study, the teachers often attributed their focus on procedural mastery to pressures associated with meeting the expectations of the course. These findings support existing literature (e.g., Brown, 2002; Grossman & Stodolsky, 1995) which acknowledge the impact of teachers' perceptions of course-related constraints such externally assessed examinations, and curriculum-related demands, on their PCK. Brown (2002) highlights the tendency for senior secondary mathematics teachers and their students to adopt surface learning approaches to mastering solution processes, based on their perception of the most appropriate way to meet the requirements of the external examination. Grossman and Stodolsky's (1995) notion of "content as context" was used to acknowledge the impact of secondary teachers' perception of the constraints of course content and the demands of meeting the requirements of the curriculum in a limited timeframe.

A key methodological contribution of the study involves the way in which the results have been structured to depict the enactment of PCK in the senior secondary mathematics classroom. The results are presented in the form of a series of scenarios illustrating rich descriptions of the teachers' PCK from the perspectives of the researcher, the teachers, and the students. The structure of the scenarios (see Section 4.1) offers a model for the presentation and analysis of detailed evidence of teaching in action and allows the different perspectives to complement and contrast each other. In addition, the presentation of descriptions of the classroom activity and interactions, before the researcher's analysis, allowed other viewers (or readers) to see what happened and, if necessary, draw their own conclusions about their observations.

6.6 Implications for Future Research

The following section discusses the implications for future research of the findings of this exploratory study. These implications relate to two key areas: the teachers' perceptions of constraints around meeting the course requirements, and possible differences in their enacted PCK for teaching probability compared to that of calculus.

The findings of the present study point to the need for further research into ways in which tensions may be reconciled between apparent limitations that relate to meeting external assessment requirements, and the value of engaging in rich mathematical thinking. In this study, the teachers' tendency to privilege procedural mastery was underpinned by their view of the expectations of the Mathematics Methods course, and the external examination. At the same time, however, the teachers alluded to the merits of exploring mathematical concepts in greater depth, but ultimately treated this as an unaffordable luxury given the content-dense nature of the

Mathematics Methods course. These findings reiterate the work of Brown (2002) who reported that the senior secondary mathematics teachers in his study believed that overcoming obstacles relating to institutional factors such as external examinations, were outside of their control.

Similarly, Leikin and Levav-Waynberg (2007) highlight a mismatch between research-based recommendations about the value of rich, conceptually focused mathematics teaching, and the reality of the classroom. Leikin and Levav-Waynberg attribute this mismatch to a lack of consistency of common beliefs and norms among stakeholders including teachers, researchers, and educational authorities, and suggest that any reforms to teaching approaches would need to stem from the broader education system and the curriculum level.

Findings of the present study support this suggestion, based on evidence including situations where the teachers expressed different mathematics-related views from each other, but made similar instructional decisions based on their perceptions of the expectations of the course. The influence of the broader course context on the teachers' PCK has implications for the extent to which they can be mathematical "with and in front of the learner" in the rich and nuanced way that Mason (e.g., 2008a) describes; that is where the teacher is sensitive to and aware of opportunities to enhance students' learning of mathematics in powerful ways. Further research into how teachers perceive "being mathematical" in the senior secondary mathematics classroom is needed in order to find out more about the ways in which this influences their interpretation and enactment the curriculum within the broader course context.

Another implication for future research relates to the extent to which PCK is topic-specific. Evidence suggests there were differences in the teachers' perceptions of the challenges associated with teaching probability compared to those relating to calculus. For example, they tended to discuss the complexities around making the content of probability meaningful for students, particularly in relation to measures of spread. At times the teachers associated limitations in their own mathematics content knowledge with these complexities, a consequence perhaps of the fact that the topic of probability is relatively new to the senior secondary mathematics curriculum in comparison to topics such as calculus. As such, further research into the topic-specific nature of teacher knowledge within the senior secondary mathematics context is needed. More specifically, it would be useful to compare, in greater depth, the enactment of teachers' PCK for teaching probability with that of topics including calculus, which have long been considered traditional aspects of senior secondary mathematics, and to consider other topics as well. Additionally, it would be illuminating to examine how PCK and pedagogical approaches vary for other senior secondary mathematics subjects where rigour, assessment, and contextual constraints vary.

6.7 Concluding Remarks

This exploratory study has highlighted some of the complexities of PCK at the senior secondary mathematics level from the perspectives of the researcher observing the classroom, the teachers, and their students. The researcher observed the enactment of multiple and interconnected aspects of PCK that focused on the processes required to solve routine text-book problems. The teachers' justification for their own instructional choices and actions reflected a perception of the constraints of the context of the Mathematics Methods curriculum, particularly in relation to the high-stakes external examination. These perceived constraints influenced the teachers' PCK in terms of the decisions they made about what to teach and how to teach it, including a tendency to avoid addressing the deeper conceptual underpinnings of mathematical ideas. The students particularly noticed and discussed those aspects of their teacher's PCK that focused on explicit demonstrations that assisted them to follow the steps involved in completing routine problems.

This study has identified some of the complexities and tensions relating to the enactment of PCK in the senior secondary mathematics classroom. In particular, there is a suggestion that classroom norms relating to the procedural demands of the "typical" exam and text-book problems are part of a teacher's PCK and can affect priorities about what is taught and attended to by teachers and students.

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Appendix A

Human Ethics Approval

Social Science Ethics Officer
Private Bag 01 Hobart
Tasmania 7001 Australia
Tel: (03) 6226 2763
Fax: (03) 6226 7148
Katherine.Shaw@utas.edu.au

HUMAN RESEARCH ETHICS COMMITTEE (TASMANIA) NETWORK

UTAS

29 April 2014

Dr Tracey Muir
Faculty of Education
Locked Bag 1307

Student Researcher: Nicole Maher

Sent via email

Dear Dr Muir

Re: FULL ETHICS APPLICATION APPROVAL
Ethics Ref: **H0013931 - Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom**

We are pleased to advise that the Tasmania Social Sciences Human Research Ethics Committee approved the above project on 26 April 2014.

This approval constitutes ethical clearance by the Tasmania Social Sciences Human Research Ethics Committee. The decision and authority to commence the associated research may be dependent on factors beyond the remit of the ethics review process. For example, your research may need ethics clearance from other organisations or review by your research governance coordinator or Head of Department. It is your responsibility to find out if the approval of other bodies or authorities is required. It is recommended that the proposed research should not commence until you have satisfied these requirements.

Please note that this approval is for four years and is conditional upon receipt of an annual Progress Report. Ethics approval for this project will lapse if a Progress Report is not submitted.

The following conditions apply to this approval. Failure to abide by these conditions may result in suspension or discontinuation of approval.

1. It is the responsibility of the Chief Investigator to ensure that all investigators are aware of the terms of approval, to ensure the project is conducted as approved by the Ethics Committee, and to notify the Committee if any investigators are added to, or cease involvement with, the project.

A PARTNERSHIP PROGRAM IN CONJUNCTION WITH THE DEPARTMENT OF HEALTH AND HUMAN SERVICES

2. Complaints: If any complaints are received or ethical issues arise during the course of the project, investigators should advise the Executive Officer of the Ethics Committee on 03 6226 7479 or human.ethics@utas.edu.au.

3. Incidents or adverse effects: Investigators should notify the Ethics Committee immediately of any serious or unexpected adverse effects on participants or unforeseen events affecting the ethical acceptability of the project.

4. Amendments to Project: Modifications to the project must not proceed until approval is obtained from the Ethics Committee. Please submit an Amendment Form (available on our website) to notify the Ethics Committee of the proposed modifications.

5. Annual Report: Continued approval for this project is dependent on the submission of a Progress Report by the anniversary date of your approval. You will be sent a courtesy reminder closer to this date. **Failure to submit a Progress Report will mean that ethics approval for this project will lapse.**

6. Final Report: A Final Report and a copy of any published material arising from the project, either in full or abstract, must be provided at the end of the project.

Yours sincerely

Katherine Shaw
Executive Officer
Tasmania Social Sciences HREC

Appendix B

Catholic Education Approval

TASMANIAN CATHOLIC education office

16 May 2014

Ms Nicole Maher
N.Maher@utas.edu.au

Dear Nicole

I am writing in regard to your recent request to conduct the research study; *Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom* involving two teachers of Mathematics Methods (MTA315) from St Patrick's College (Launceston) and their students.

I have read the information provided which outlines details of this research project and, subsequently, I am happy to provide in principle approval. Please note however, that it is up to the St Patrick's College teachers to determine whether they wish to participate in the study.

Please do not hesitate to contact this office if you require any further information.

Yours sincerely

Dr Trish Hindmarsh
Director

th kk

The D'Arcy Centre PO Box 102 North Hobart 7002 Tel: 03 6210 8888 State Office & Regional Office: South
MacKillop Centre PO Box 160 Riverside 7250 Tel: 03 6327 1795 Regional Office: North
Tenison Woods Centre PO Box 564 Ulverstone 7315 Tel: 03 6425 1603 Regional Office: North West
ceo@catholic.tas.edu.au www.catholic.tas.edu.au Fax: 03 6210 8844

Appendix C
Preamble Email to Principals

Dear,

My name is Nicole Maher, I am a PhD candidate at the University of Tasmania, under the supervision of Dr Tracey Muir and Associate Professor Helen Chick. I am writing to request your permission to invite some of your experienced teachers of Mathematics Methods 3 (MTA315), and their students, to participate in my research project.

My study aims to investigate the kind of knowledge that effective mathematics teachers have, in order to convey complex mathematics ideas to their students. Such knowledge encompasses more than the knowledge of the mathematics content itself, and I am particularly interested in those aspects of this knowledge that students identify as being particularly beneficial for their own learning. I fully appreciated that Mathematics Methods 3 is a demanding pre-tertiary syllabus, and you can be assured that your teachers' and students' participation in this study would not interfere with the usual activities of teaching and learning. Data collection would occur at mutually agreed times and not around revision and examination periods.

Data collection, including lesson observation and interviews, would involve up to 8 lessons in total per Mathematics Methods class, over two different times of the year. I have attached the letters of introduction and consent forms for both teachers and students which explain, in detail, the procedures involved in participating in this study. Thank you for taking the time to consider allowing your school to assist with this study and please do not hesitate to contact me if you require any further clarification. Would you please advise me of your decision to participate by return email?

Kind regards,

Nicole Maher

Appendix D
Teacher Information Sheet

Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom

Teacher Information Letter

Dear Teacher,

1. Invitation
You are invited to participate in a study to explore the teaching and learning of senior secondary mathematics content in MTA315 Mathematics Methods. This study is being conducted in partial fulfilment of a PhD for Nicole Maher, under the supervision of Dr Tracey Muir and Associate Professor Helen Chick.

2. What is the purpose of this study
The purpose of this study is to investigate the ways in which teachers transform the content of senior secondary mathematics to make it comprehensible to their students, and to explore this from the perspectives of both teachers and students. We are particularly interested in those actions carried out by teachers that students identify as being particularly helpful for their learning of the content in this course, such as explanations, specific examples, and diagrams. The personal qualities and attributes of individual teachers will not be the focus of this research

3. Why have I been invited to participate?
You have been selected to participate in this study because you are currently teaching a senior secondary mathematics course, MTA315 Mathematics Methods, and have the experience to be able to talk about the instructional decisions that you make in the classroom.

4. What will I be asked to do?
If you consent to participate in this study, you will be invited to contribute data in the following ways:

- by having your teaching observed and video-recorded and your written work (e.g., worked examples, diagrams or explanations on whiteboard or on paper) photographed.
- by participating in audio-recorded follow-up interviews for up to 20 minutes after each lesson.

You will also be asked to distribute consent forms to students and parents. Details of the above activities are given in the next sections.

Lesson observations

The researcher will observe up to 8 double lessons of your MTA315 Mathematics Methods class. It is anticipated that these 8 lessons will include at least two different mutually agreed topic areas, preferably contrasting areas such as probability and function study. The lesson observations will be as unobtrusive as possible and will be conducted by the researcher while you and your class are engaging in usual activities of teaching and learning. Data collected during the lesson observations will be in the form of field notes that document the teaching and learning interactions that take place between yourself and participating students, as well as aspects of your direct instruction to the whole class. During the last 10 minutes of each lesson, with your permission, some consenting students will be invited to complete a short answer questionnaire relating to their learning in the day's lesson.

Video recording of lessons and written work photographed
With your consent and that of at least some of your students, and their parents, the lessons observed by the researcher will be video-recorded. The primary purpose of video-recording each lesson is to obtain documentation of teaching and learning interactions for later description. The researcher may also take photographs of participants' written work such as your examples on the whiteboard, and students' responses to mathematical tasks in their exercise books to supplement the video footage.

The video camera will be placed in a fixed position at the back of the classroom, with the lens set on a wide-angle in view of all participating students and yourself. Students who have not consented) will be seated in an area of the classroom that is out of video shot range. If any of these students inadvertently appear in any of the recordings they will be pixelated in the recordings. Only the researchers and possibly, on rare occasions yourself, will see the video footage.

Audio recorded interviews
After each of the eight lessons observed by the researcher, you will be invited to participate in an interview with the researcher at a mutually convenient time. Each interview will take no longer than 20 minutes and will be audio-recorded and transcribed. During the interviews the researcher will invite you to respond to questions that arise from the following sources:

- Observations of particular teaching and learning interactions that occurred during the day's lesson. On rare occasions, this may include viewing video footage, as a visual reminder of particular learning and teaching events.
- General questions about your own identification and perception of significant teaching and learning interactions.

You will be offered the option to read and amend the transcripts of your own interviews.

5. Are there any possible benefits from participation in this study?
The study will give you an opportunity to reflect upon, examine and discuss your own practice. The mathematics education research community and the teaching community may benefit from the findings of this study in terms of identifying the kinds of teaching practices that are most influential in assisting students in their learning of senior secondary mathematics content. The findings from this study will offer teachers, researchers and students some further insight into how mathematics content at this abstract level is transformed in ways that are most beneficial to students' learning from their perspectives.

6. Are there any possible risks from participation in this study?
The researchers appreciate that MTA315 Mathematics Methods is an externally assessed pre-tertiary subject and so your participation in the study will not interfere with the usual activities of teaching and learning with your class. Data collection will occur at mutually agreed times and not around revision and examination periods. Although this is not anticipated there is a chance that you may feel anxious during an interview or during lesson observations when your teaching is being observed and video recorded, and aspects of your written work photographed. During the interviews you can decline to answer any or all questions or ask that the interview cease at any time without any explanation or consequence. Similarly you may ask that any observation and video recording and photographing of your participation in the lesson cease at any time without explanation or consequence.
You will be able to view and amend interview transcripts and ask that any unprocessed part of the data or all unprocessed data that you have contributed be withdrawn from the study at any point during the project. If you experience any discomfort as a result of any aspect of this research you are able to access free counselling provided through the Department of Education by calling Vicki Martin and Associates on free call 1800 064 039.

7. What if I change my mind during or after the study?
If you decide to decline your participation at any time, you may do so without providing an explanation. You will be able to view and amend your own interview transcripts and ask that any unprocessed part of the data or all unprocessed data that you have contributed be withdrawn from the study at any point during the project.

8. What will happen to the information when this study is over?
Surveys, hard copies of interview transcripts, audio and video files, and photographs will be stored on the Launceston campus of the University of Tasmania in locked cabinet in room A013, accessible only by the researchers. Your name and other identifying information will be removed from these data and replaced with a code. Computer files will be password protected and stored on a secure server at the Faculty of Education, Launceston campus. After a period of five years from the publication of the thesis, all transcripts and field notes will be shredded, computer files delete, raw audio and video recordings, and photographs deleted. All information collected by the researchers will be treated confidentially. We will remind all participants of the importance of confidentiality but cannot guarantee that other participants, such as students participating in focus groups, will maintain confidentiality.

9. How will the results of the study be published?
After the completion of data collection at the end of 2014, the researcher will provide a summary report of the data for participating teachers and students. You will be provided with the thesis in electronic form by the end of the 2016 school year. The thesis will also be available to students and their parents upon request. Teachers, students and schools will be anonymous in all publication of results. Pseudonyms will be used when referring to quotes from interview transcripts and in descriptions from lesson observations in all publications of results of the study.

10. What if I have questions about this study?

If you have any questions relating to this study, please feel free to contact one of the researchers:

Dr Tracey Muir: University of Tasmania (Launceston)
Telephone: 6324 3261, email: Tracey.Muir@utas.edu.au

Associate Professor Helen Chick: University of Tasmania (Hobart)
Telephone: 6226 7220, email: Helen.Chick@utas.edu.au

Nicole Maher: University of Tasmania (Launceston)
Email: Nicole.maher@utas.edu.au

This study has been approved by the Tasmanian Social Sciences Human Research Ethics Committee. If you have concerns or complaints about the conduct of this study, please contact the Executive Officer of the HREC (Tasmania) Network on +61 3 6226 7479 or email human.ethics@utas.edu.au. The Executive Officer is the person nominated to receive complaints from research participants. Please quote ethics reference number H0013931

Thank you for taking the time to consider this research. If you would like to participate in this study, please indicate on the consent form, the aspects of the research in which you agree to be involved and sign it. Please place your consent form in the envelope provided and hand it to the school office where the researcher will collect it. This information sheet is for you to keep.

Appendix E

Teacher Consent Form

Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom

Teacher Consent Form

1. I have read and understood the Information Sheet for this study.
2. The nature and possible effects of the study have been explained to me.
3. I understand that the study involves:
    - Having my teaching observed by the researcher for up to 8 of my MTA315 Mathematics Methods lessons.
    - Having my teaching video-recorded by the researcher for up to 8 of my MTA315 Mathematics Methods lessons.
    - Having photographs taken of my written work that I produce/use in class.
    - Participating in a post-lesson audio recorded interview following each lesson observed by the researcher.
4. I understand that my participation in this study involves low risk.
5. I understand that all research data will be securely stored on the Launceston campus of the University of Tasmania.
6. Any questions that I have asked have been answered to my satisfaction.
7. I understand that the researcher(s) will maintain confidentiality and that any information that I supply to the researcher(s) will be used only for the purposes of the research. I understand that in any public documents arising from this research, pseudonyms will be used for my own name and the names of my school and students.
8. I understand that the results of the study will be published so that I cannot be identified as a participant.
9. I understand that my participation is voluntary and that I may withdraw at any time without any effect.
   If I so wish, I may request that any unprocessed data I have supplied be withdrawn from the research.

I give consent to participate in this study.
Yes ☐ No ☐

Participant's name:
____

Participant's signature:
____

Date: ________________________

Statement by Investigator

☐ I have explained the project and the implications of participation in it to this volunteer and I believe that the consent is informed and that he/she understands the implications of participation.

If the Investigator has not had an opportunity to talk to participants prior to them participating, the following must be ticked.

☐ The participant has received the Information Sheet where my details have been provided so participants have had the opportunity to contact me prior to consenting to participate in this project.

Investigator's name:
____

Investigator's signature:
____

Date: ________________________

Appendix F

Parent Information Letter

Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom

Parent/Care giver Information Sheet

Dear Parent/Guardian

1. Invitation

Your child is invited to participate in a study to explore the teaching and learning of senior secondary mathematics content in MTA315 Mathematics Methods. This study is being conducted in partial fulfilment of a PhD for Nicole Maher under the supervision of Dr Tracey Muir and Associate Professor Helen Chick.

2. What is the purpose of this study?

The purpose of this study is to explore the way in which your child's teacher helps students to learn senior secondary mathematics. Teachers use a range of approaches to express and present mathematics content in ways that assist students to learn particular skills and concepts, including explanations, examples, diagrams and there are many more. We are particularly interested in those actions carried out by your child's teacher that he/she identifies as being particularly helpful for his/her learning of the content in MTA315 Mathematics Methods. Your child's perspective will provide valuable insight into teachers', researchers' and other students' understanding of what are the most powerful ways in which teachers transform senior secondary mathematics content to enable students to grasp it.

3. Why has your child been invited to participate?

Your child has been selected to participate in this investigation because he/she is studying MTA15 Mathematics Methods, and his/her teacher has also been selected to participate.

The researchers appreciate that MTA315 Mathematics Methods is one of your child's pre-tertiary subjects and that this is a busy year for him/her. Your child's participation or non-participation will in no way interfere with his/her school commitments. Furthermore, his/her decision to participate or not to participate in this study will in no way impact upon any aspect of his/her enrolment in MTA315 Mathematics Methods.

4. What will your child be asked to do?

If you and your child consent to your child's participation in this study, he/she will be invited to contribute data in the following ways:

- by being part of the class that will be observed and video recorded by the researcher.
- by having some of his/her written work photographed.
- by participating in audio-recorded focus groups for up to 20 minutes after each lesson.
- by completing a post lesson short-answer questionnaire after each lesson.

Further details of each of the above activities are given in the following sections.

Lesson observations

The researcher will observe up to 8 of your child's MTA315 Mathematics Methods lessons. These lessons will not necessarily be consecutive but will ideally include lessons on at least two different topic areas. If you do not want your child to be involved in this part of the research then he/she can still attend the lessons as usual but the researcher will not observe or take notes on any aspect of your child's involvement in the lesson. You are not expected to do anything differently; the class will be conducted in the usual ways.

Lessons video recorded and written work photographed

With your consent and that of your child and his/her teacher, the lessons observed by the researcher will be video-recorded. The video camera will be placed in a fixed position at the back of the classroom, with the lens set on a wide-angle in view of all consenting students and the teacher. If you do not give consent for your child to be video-recorded, he/she may still attend the mathematics lessons as usual, but the researcher will make sure that your child is not within video shot range. If inadvertently your child does appear in any video footage, his/her image will be pixelated.

In order to enhance the detail of aspects of the video footage, the researcher may also take photographs of participants' written work, such as the teacher's examples on the whiteboard, and consenting students' responses to mathematical tasks written into their books. Photographs of your child's work will not be taken without his/her and your consent.

Post-lesson questionnaire

During the last part of each lesson observed by the researcher, your child will be invited to complete a post-lesson questionnaire. The questionnaire should take no more than 10 minutes to complete and will involve answering two questions relating to your child's learning in the day's lesson

Post-lesson interview

To help us to better understand the way that teaching affects students' learning, we would like to be able to talk to your child about what worked well for them, in the lesson we observed. If you give consent, your child may be invited to participate, along with up to 5 other student from his/her class, in an audio-recorded focus group after each lesson that is observed by the researcher. The focus groups will take no longer than 20 minutes and will take place at a mutually suitable time as soon after the lesson as feasible. This may involve part of your child's lunch-time or part of a study line. During the focus groups, your child will be invited to respond to questions that arise from the researcher's observations of particular teaching and learning interactions during the lesson itself. Your child may also be invited to respond to questions in relation to his/her own responses to the post-lesson questionnaire. The researcher will ask for your child's permission to share his/her responses before referring to these in the focus groups.

Your child will be offered the option of reading the transcripts of his/her own focus groups. Some of his/her focus group responses that refer to teaching actions that assisted him/her with his/her learning, may be shared with the teacher. Your child's individual identify however, will not be disclosed at any time.

Audio recordings of the interviews will be heard only by the researchers and pseudonyms will be used to label files containing transcripts or summaries. Audio files will be stored in password protected digital audio files on a secure server at the University of Tasmania, Launceston Campus.

Your child will be asked to respect the confidentiality of all other participants and not to disclose any information shared during the focus groups.

Your informed consent

If you wish your child to participate in this study you will be asked to provide separate consent to each of these components of the research. You may give consent for your child to contribute to some, all or none of the components of this research.

5. Are there any possible benefits from participation in this study?

Participation in this study will give your child the opportunity to reflect on his/her learning of the content that he/she is studying in mathematics and to identify aspects of teaching practice that particularly assist him/her with his/her learning.

The mathematics education research community and the teaching community may benefit from the findings of this study in terms of identifying the kinds of teaching practices that are most influential in assisting students in their learning of senior secondary mathematics content.

6. Are there any possible risks from participation in this study?

Although this is not anticipated there is a chance that your child may feel anxious during an interview or while he/she is participating in a lesson that is being observed and video recorded. During the interviews your child can decline to answer any or all questions or ask that the interview cease at any time without any explanation or consequence. Similarly your child may ask that any observation and video-recording and photographing of his/her participation in the lesson cease at any time without explanation or consequence If your child experiences discomfort as a result of any aspect of the research you are able to access free counselling provided through the Department of Education by calling Vicki Martin and Associates on free call 1800 064 039.

7. What if I change my mind during or after the study?

If you decide to withdraw your child's participation at any time, you may do so without providing an explanation.

8. What will happen to the information when the study is over?

Surveys, hard copies of interview and focus group transcripts, audio and video files, and photographs will be stored on the Launceston campus of the University of Tasmania in locked cabinet in A013 accessible only by the researchers. Your child's name and other identifying information will be removed from these documents. Computer files will be password protected and stored on a secure server at the Faculty of Education, Launceston campus. No sooner than 5 years from the publication of the PhD thesis, all transcripts and field notes will be shredded and computer files deleted. All information collected by the researchers will be treated confidentially. We will remind all participants of the importance of confidentiality but cannot guarantee that other participants will maintain confidentiality.

9. How will the results of the study be published?
After the completion of data collection at the end of 2014, the researcher will provide a summary report of the data for participating teachers and students. Participating schools and teachers will be provided with the thesis in electronic form by the end of the 2016 school year. The thesis will also be available to students and their parents upon request. Your child, his/her teacher and your child's school will be anonymous in all publication of results. Pseudonyms will be used when referring to quotes from interview transcripts and in descriptions from lesson observations in all publications of results of the study.

10. What if I have questions about this study?
If you have any questions relating to this study, please feel free to contact one of the researchers:
Dr Tracey Muir: University of Tasmania (Launceston)
Telephone: 6324 3261, email: Tracey.Muir@utas.edu.au

Associate Professor Helen Chick: University of Tasmania (Hobart)
Telephone: 6226 7220, email: Helen.Chick@utas.edu.au

Nicole Maher: University of Tasmania (Launceston)
Email: Nicole.maher@utas.edu.au

This study has been approved by the Tasmanian Social Sciences Human Research Ethics Committee. If you have concerns or complaints about the conduct of this study, please contact the Executive Officer of the HREC (Tasmania) Network on +61 3 6226 7479 or email human.ethics@utas.edu.au. The Executive Officer is the person nominated to receive complaints from research participants. Please quote ethics reference number

Thank you for taking the time to consider this research. There are two consent forms attached, one is yours and one is for your child. The consent forms include all options for participation in the study. If you would like your child to take part in the research please sign your consent form and indicate which option(s) you would like your child to be involved in. When both you and your child have completed both consent forms, please place them in the envelope provided, seal it and ask your child to hand it to his/her MTA315 Mathematics Teacher for the researcher to collect. This information sheet is for you to keep.

Appendix G

Student Information Letter

Student and teacher perspectives on pedagogical content knowledge in the senior secondary mathematics classroom

Student Information Sheet

Dear Student

1. Invitation

You are invited to participate in a study to explore the teaching and learning of senior secondary mathematics content in MTA315 Mathematics Methods. This study is being conducted in partial fulfilment of a PhD for Nicole Maher under the supervision of Dr Tracey Muir and Associate Professor Helen Chick.

2. What is the purpose of this study?

The purpose of this study is to explore the way your teacher helps you to understand the content of senior secondary mathematics. Teachers use a range of approaches to express and present mathematics content in ways that assist you to learn particular skills and concepts, including their use of explanations and examples, diagrams and there are many more. We are particularly interested in those actions carried out by your teacher that you identify as being particularly helpful for your learning of the content in this course. Your perspective will provide valuable insight into teachers', researchers' and other students' understanding of what are the most powerful ways in which teachers transform senior secondary mathematics content to enable students to grasp it.

3. Why have I been invited to participate?

You have been selected to participate in this investigation because you are studying an intellectually challenging senior secondary course in mathematics (MTA315 Mathematics Methods) and your teacher has also been selected to participate.
The researchers appreciate that MTA315 Mathematics Methods is one of your pre-tertiary subjects and that this is a busy year for you. We can assure you that your participation will in no way interfere with your school commitments.

4. What will I be asked to do?

If you and your parents' consent to your participation in this study, you will be invited to contribute data in the following ways:

- by being part of the class that will be observed and video-recorded by the researcher.
- By having some of your written work photographed by the researcher.
- by completing post lesson short-answer questionnaires
- by participating in audio-recorded focus group interviews for up to 20 minutes after each lesson.

Details of each of the above activities are given in the following sections.

Lesson observations

The researcher will observe up to 8 of your MTA315 Mathematics Methods lessons. The lesson observations will be unobtrusive and will be conducted by the researcher while you are engaging in usual activities of learning in the classroom. If you do not wish to be involved in this part of the research you may still attend your mathematics lessons as usual but the researcher will not observe or take notes on any aspect of your involvement in the lesson. You are not expected to do anything differently in the lessons; the class will be conducted in the usual ways.

Lessons video recorded and written work photographed

With your consent and that of your teacher and some other students, the lessons observed by the researcher will be video-recorded. If you do not give consent to be video-recorded, you may still attend the mathematics lessons as usual, and the researcher will make sure that you are not within video shot range. If inadvertently you do appear in any video footage, your image will be pixelated. In order to supplement the detail of the video footage, the researcher may also ask for your permission to take photos of some of your mathematics work.

Post-lesson questionnaire

During the last part of each lesson observed by the researcher, you will be invited to complete a post-lesson questionnaire. The questionnaire should take no more than 10 minutes to complete and will involve answering two questions relating to your learning in the day's lesson.

Post-lesson interview

To help us better understand the way that teaching affects your learning, we would like to be able to talk to you about what worked well for you in the lessons that we observed. If you are willing, you may be invited to participate, along with up to 5 other student from your class, in an audio-recorded focus group after each lesson that is observed by the researcher. The focus groups will take no longer than 20 minutes and will take place at a mutually suitable time as soon after the lesson as feasible. This may involve part of your lunch-time or study line. During the focus groups, you will be invited to respond to questions that arise from the researcher's observations of particular teaching and learning interactions during the lesson itself. You may also be invited to respond to questions in relation to your own responses to the post-lesson questionnaire. The researcher will ask for your permission to share your responses before referring to these in the focus groups.

You will be offered the option of reading the transcripts of your own focus groups interviews.

Audio recordings of the focus group interviews will be heard only by the researchers and pseudonyms will be used to label files containing transcripts or summaries of these

You will be asked to respect the confidentiality of all other participants and not to disclose any information shared during the focus groups.

Your informed consent

If you wish to participate in this study you will be asked to provide separate consent to each of these components of the research. You may give consent to contribute to some, all or none of the components of this research.

5. Are there any possible benefits from participation in this study?

Participation in this study will give you the opportunity to reflect on your learning of the content that you are studying in mathematics and to identify aspects of teaching practice that particularly assist you with your learning.

The mathematics education research community and the teaching community may benefit from the findings of this study in terms of identifying the kinds of teaching practices that are most influential in assisting students in their learning of senior secondary mathematics content.

6. Are there any possible risks from participation in this study?

Although this is not anticipated there is a chance that you may feel anxious during a focus group interview or while you are participating in a lesson that is being observed and video recorded. During the focus group interviews you can decline to answer any or all questions or ask that your participation in the focus group cease at any time without any explanation or consequence. Similarly, you may ask that any observation and video recording and photographing of your participation in the lesson cease at any time without explanation or consequence.

You will be able to view and amend interview transcripts and ask that any unprocessed part of the data or all unprocessed data that you have contributed be withdrawn from the study at any point during the project. If you experience any discomfort as a result of any aspect of this research you are able to access free counselling provided through the Department of Education by calling Vicki Martin and Associates on free call 1800 064 039.

7. What if I change my mind during or after the study?

If you decide to decline your participation at any time, you may do so without providing an explanation.

8. What will happen to the information when this study is over?

Surveys, hard copies of interview transcripts and audio and video files, will be stored on the Launceston campus of the University of Tasmania in locked cabinet in the office of Nicole Maher, room A013 and will be accessible only by the researchers. Your name and other identifying information will be removed from these documents. Computer files will be password protected and stored on a secure server at the Faculty of Education, Launceston campus. No sooner than 5 years from the publication of the PhD thesis, all transcripts and field notes will be shredded and computer files deleted. All information collected by the researchers will be treated confidentially. We will remind all participants of the importance of confidentiality but cannot guarantee that other participants will maintain confidentiality.

9. How will the results of the study be published?

After the completion of data collection at the end of 2014, the researcher will provide a summary report of the data for participating teachers and students. Your school will be provided with the thesis in electronic form by the end of the 2016 school year. The thesis will also be available to you and your parents upon request. You, your teacher and your school will be anonymous in all publication of results. What if I have questions about this study?

If you have any questions relating to this study, please feel free to contact one of the researchers:

Dr. Tracey Muir: University of Tasmania (Launceston)
Telephone: 6324 3261, email: Tracey.Muir@utas.edu.au

Associate Professor Helen Chick: University of Tasmania (Hobart)
Telephone: 6226 7220, email: Helen.Chick@utas.edu.au

Nicole Maher: University of Tasmania (Launceston)
Email: Nicole.maher@utas.edu.au

This study has been approved by the Tasmanian Social Sciences Human Research Ethics Committee. If you have concerns or complaints about the conduct of this study, please contact the Executive Officer of the HREC (Tasmania) Network on +61 3 6226 7479 or email human.ethics@utas.edu.au. The Executive Officer is the person nominated to receive complaints from research participants. Please quote ethics reference number H0013931.

Thank you for taking the time to consider this research. There are two consent forms attached, one is yours and one is for your parents as they also need give their consent for you to participate in this study. The consent forms include all options for participation in the study. If you wish to take part in the research please sign your consent form and indicate which option(s) you wish to be involved in. When both you and your parents have completed both consent forms, please place them in the envelope provided, seal it and hand it to your MTA315 Mathematics Teacher for the researcher to collect. This information sheet is for you to keep if you wish.

Appendix H

Parent Consent Form

Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom

Parent Consent Form

1. I have read and understood the Information Sheet for this study.
2. The nature and possible effects of the study have been explained to me.
3. I understand that the study involves:
   - Having my child's involvement in up to 8 of his/her MTA315 Mathematics Methods lessons observed by the researcher.

     I give consent for my child's involvement in these lessons to be observed.

     Yes ☐ No ☐
   - Having my child's involvement in up to 8 of his/her MTA315 Mathematics Methods lessons video recorded by the researcher.

     I give consent for my child's involvement in these lessons to be video recorded.

     Yes ☐ No ☐
   - Photographs taken of some of my child's written work.

     I give consent for my child's written work to be photographed.

     Yes ☐ No ☐
   - My child completing a 10 minute short-answer questionnaire at the end of each of his/her lessons observed by the researcher.

     I give consent for my child to complete the 10 minute questionnaires at the end of these lessons.

     Yes ☐ No ☐
   - My child participating in a 20 minute post-lesson, audio- recorded focus group after each lesson that is observed by the researcher. This focus group may involve up to 5 other students from my child's maths class.

     I give consent for my child to participate in the 20 minute post-lesson focus groups.

     Yes ☐ No ☐

     I give consent for my child's focus group responses to be audio-recorded by the researcher and agree that my child must keep the discussion from focus group interviews confidential.

Yes ☐ No ☐

4. I understand that my child's participation in this study involves low risk.
5. I understand that all research data will be securely stored on the Launceston campus of the University of Tasmania.
6. Any questions that I have asked have been answered to my satisfaction.
7. I understand that the researcher(s) will maintain confidentiality and that any information that my child supplies to the researcher(s) will be used only for the purposes of the research. I understand that the researchers will remind participants of the importance of confidentiality but cannot guarantee that other participants will maintain confidentiality such as when several participants are involved in a focus group.
8. I understand that the results of the study will be published so that my child cannot be identified as a participant.
9. I understand that my child's participation is voluntary and that he/she may withdraw at any time without any effect.

   If I so wish, I may request that any unprocessed data my child has supplied be withdrawn from the research.

Participant's name:
____

Participant's signature:
____

Date: ________________________

Statement by Investigator

☐ I have explained the project and the implications of participation in it to this volunteer and I believe that the consent is informed and that he/she understands the implications of participation.

If the Investigator has not had an opportunity to talk to participants prior to them participating, the following must be ticked.

☐ The participant has received the Information Sheet where my details have been provided so participants have had the opportunity to contact me prior to consenting to participate in this project.

Investigator's name:
____

Investigator's signature:
____

Date: ________________________

Appendix I

Student Consent Form

Student and teacher perspectives on pedagogical content knowledge in the senior mathematics classroom

Student Consent Form

1. I have read and understood the Information Sheet for this study.
2. The nature and possible effects of the study have been explained to me.
3. I understand that the study involves:

   - Having my involvement in up to 8 of my MTA315 Mathematics Methods lessons observed by the researcher.

     I agree to have my involvement in these lessons observed.

     Yes ☐ No ☐
   - Having my involvement in up to 8 of my MTA315 Mathematics Methods lessons video recorded by the researcher.

     I agree to have my involvement in these lessons video recorded.

     Yes ☐ No ☐
   - Having some of my written work photographed during the lessons by the researcher.
   - I agree to have my written work photographed by the researcher.

     Yes ☐ No ☐
   - Completing a short-answer questionnaire at the end of each of the lessons observed by the researcher.

     I agree to complete the questionnaires at the end of these lessons.

     Yes ☐ No ☐
   - Participating in a post-lesson audio recorded focus group interviews with up to 5 other students in my class, after each lesson that is observed by the researcher.

     I agree to participate in the post-lesson focus group interviews.

     Yes ☐ No ☐

     I agree to have my responses in the focus group interviews audio-recorded by the researcher and I agree to keep the discussion from the focus group interviews confidential.

     Yes ☐ No ☐

4. I understand that my participation in this study involves low risk.
5. I understand that all research data will be securely stored on the Launceston campus of the University of Tasmania.
6. Any questions that I have asked have been answered to my satisfaction.
7. I understand that the researcher(s) will maintain confidentiality and that any information I supply to the researcher(s) will be used only for the purposes of the research. I understand that the researchers will remind participants of the importance of confidentiality but cannot guarantee that other participants will maintain confidentiality such as when several participants are involved in a focus group interview. I understand that any public documents arising from this research will use pseudonyms for my name, the name of my school and my teacher.
8. I understand that the results of the study will be published so that I cannot be identified as a participant.
9. I understand that my participation is voluntary and that I may withdraw at any time without any effect.

   If I so wish, I may request that any unprocessed data I have supplied be withdrawn from the research.

Participant's name: ____

Participant's signature: ____

Date: ________________________

Statement by Investigator

☐ I have explained the project and the implications of participation in it to this volunteer and I believe that the consent is informed and that he/she understands the implications of participation.

If the Investigator has not had an opportunity to talk to participants prior to them participating, the following must be ticked.

☐ The participant has received the Information Sheet where my details have been provided so participants have had the opportunity to contact me prior to consenting to participate in this project.

Investigator's name: ____

Investigator's signature: ____

Date: ________________________